Gelfand duality for compact pospaces

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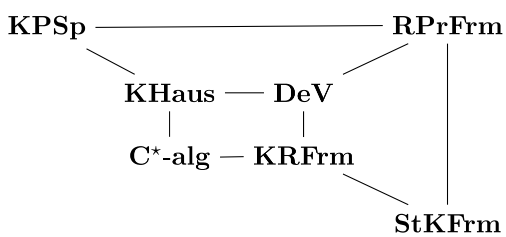
Let X be a compact Hausdorff space. It is well known that X can be characterized by its ring of real continuous functions, by its lattice of regular open subsets or more simply by its frame of open subsets. These characterizations lead to dualities between the category **KHaus**, of compact Hausdorff space and respectively the categories **C*-alg**, of commutative $C^\star$-algebras, **DeV** of de Vries algebras and **KRFrm** of compact regular frames. We thus get a square of dualities.

Later, G.Bezhanishvili and J.Harding extended a part square to dualities between the categories **KPSp** of compact pospaces, **RPrFrm** of regular proximity frames and **StKFrm** of stably compact frames.

We thus get the square of dualities extended this way.



Our aim is to complete the outside triangle, looking for a category generalizing the $C^\star$-algebras.

An essential fact, due to G.Hansoul, leads us to consider a category of ordered semi-ring. Indeed, we can see that the Nachbin-Stone-Cech compactification of a completely regular ordered po-space X can be realized through its semi-ring of increasing, continuous and real, positive functions, denoted $I(X, \mathbb{R}^+)$.

Definition 1.

1. An ℓ-semi-ring is an algebra $(A, +, ., 0, 1, \leq)$ with the following axioms :
   (a) $(A, +, 0)$ and $(A, ., 1)$ are commutative monoids.
   (b) $(A, +, .)$ is distributive.
   (c) $a \leq b \Leftrightarrow a + c \leq b + c$.
   (d) $a \geq 0$ and $a \leq b \Rightarrow a.c \leq b.c$
   (e) $(A, \leq)$ is a lattice.
2. An ℓ-semi-ring A is bounded if for all $a \in A$, there is $n \in \mathbb{N}$ such that $a \leq n.1$.
3. An ℓ-semi-ring A is Archimedean if for all $a, b, c, d \in A$, whenever $n.a + b \leq n.c + d$, then $a \leq c$.
4. An ℓ-semi-ring A is an ℓ-semi-algebra if it is an $\mathbb{R}^+$-algebra such that for all $a, b \in A$ and $r \in \mathbb{R}^+$, $r.a \leq r.b$.
5. $(a \vee b) + c = (a + c) \vee (b + c)$ and $(a \wedge b) + c = (a + c) \wedge (b + c)$.

We now denote **sbal** the category of bounded Archimedean ℓ-semi-algebras, and defining the morphisms in the natural way.

Theorem 2. *There is a dual equivalence between the categories* **usbal***, of complete* **sbal***, and* **KPSp**.