On multiplicative derivations of commutative residuated lattices

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For a theory of algebras with two operations $+$ and $\cdot$, we have an interesting method, *derivations*, to develop the structure theory, as an analogy of derivations of analysis. The notion of derivation of algebras was firstly applied to the theory of ring, and then it was also applied to other algebras, such as lattices and MV-algebras. We here applied the derivation theory to the (commutative) residuated lattices which are very basic algebras corresponding to fuzzy logic. Let $L = (L, \wedge, \vee, \to 0, 1)$ be a (commutative) residuated lattice and $B(L)$ be the set of all complemented elements of L. A map $d : L \to L$ is called a *multiplicative derivation* (or simply *derivation* here) of L if it satisfies the condition

$$d(x \wedge y) = (dx \odot y) \vee (x \odot dy) \quad (\forall x, y \in L)$$

A derivation d is called *ideal* if $x \leq y$ then $dx \leq dy$ and $dx \leq x$ for all $x, y \in L$. Moreover, a derivation d is said to be *good* if $d1 \in B(L)$. For a derivation d of L, we consider a subset $\mathrm{Fix_d}(L) = \{x \in L \mid dx = x\}$ of the set of all fixed elements of L for d.

Proposition 1. *For a good ideal derivation d, we have* $\mathrm{Fix_d}(L) = d(L)$.

Theorem 1. $\mathrm{Fix_d}(L) = (\mathrm{Fix_d}(L), \wedge, \vee, \odot, \mapsto, 0, d1)$ *is a residuated lattice, where operations on* $\mathrm{Fix_d}(L)$ *are defined as follows:*

$$\begin{aligned} dx \wedge dy &= d(x \wedge y) & \qquad dx \vee dy &= d(x \vee y) \\ dx \odot dy &= d(x \odot y) & \qquad dx \mapsto dy &= d(dx \to dy). \end{aligned}$$

A filter F is called a *d-filter* if $x \in F$ implies $dx \in F$ for all $x \in L$. It is easy to show that a quotient structure L/F is also a residuated lattice for a filter F. Moreover, we have the following.

Proposition 2. *Let d be a good ideal derivation and F be a d-filter of L. A map $d/F : L/F \to L/F$ defined by $(d/F)(x/F) = dx/F$ for all $x/F \in L/F$ is a good ideal derivation of L/F.*

Therefore, the quotient structure $(d/F)(L/F) = \mathrm{Fix_{d/F}}(L/F)$ is a residuated lattice. Since F is a d-filter of L, $d(F)$ is also a filter of $d(L)$ and thus $d(L)/d(F)$ forms a residuated lattice. It is natural to ask what the relation between two residuated lattices $(d/F)(L/F)$ and $d(L)/d(F)$ is. Next result is an answer.

Theorem 2. *Let d be a good ideal derivation and F be a d-filter of L. Then we have $(d/F)(L/F) = \mathrm{Fix_{d/F}}(L/F)$ is isomorphic to $d(L)/d(F)$, that is,*

$$(d/F)(L/F) = \mathrm{Fix_{d/F}}(L/F) \cong d(L)/d(F).$$

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