Approximation algebras of Bernoulli distributions

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The problems we consider originate from theoretical computer science, yet may be formulated in algebraic terms and may eventually be of interest by themselves. Suppose we have a supply of independent Bernoulli random variables with given distributions and we want to obtain random variables with other distributions by transforming the initial ones using deterministic functions from a certain set. The question is, what kind of distributions may be obtained thus, given the initial distributions and the set of transforming functions?

The algebraic setting for the problem is the following. Let $E = \{0, 1\}$ and $\langle E, B\rangle$ be some algebra. The set $\mathbf{S}$ of all Bernoulli distributions on E is isomorphic to the segment $[0; 1]$, hence every distribution may be considered as just a number in the segment (for the sake of convenience we take this number to be the probability of 1).

For a set of mutually independent Bernoulli random variables $X_1, \ldots, X_n$ with distributions $p_1, \ldots, p_n \in \mathbf{S}$ respectively and an n-ary operation $f \in B$, the value of $f(X_1, \ldots, X_n)$ is a Bernoulli random variable as well, and its distribution may be expressed as a polylinear function of $p_1, \ldots, p_n$ that we shall denote by $\hat{f}(p_1, \ldots, p_n)$. Thus every algebra $\langle E, B\rangle$ induces an algebra of probability distributions $\langle \mathbf{S}, \hat{B}\rangle$, where $\hat{B} = \{\hat{f} \mid f \in B\}$ is a set of polylinear functions on the segment $[0; 1]$.

The random variables that may be generated by deterministic operations from B over random variables with distributions from an initial set G are exactly the subalgebra of $\langle \mathbf{S}, \hat{B}\rangle$ generated by $G \subseteq \mathbf{S}$.

We consider special subalgebras, which, besides being closed under operations from $\hat{B}$, are also topologically closed, i.e. contain all their limit points. In terms of the initial problem that corresponds to considering both distributions that may be generated and those that may be approximated with arbitrary precision. These algebras are called approximation algebras of Bernoulli distributions. The approximation algebra generated by G is defined as the intersection of all approximation algebras that contain G.

We present an overview and some very recent results on finitely generated approximation algebras of Bernoulli distributions.

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