Cyclic Extensions of Finite Groups

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Understanding the structure of groups via products of their subgroups is one of the key goals of group theory since its beginnings. The particular case when a group G is a complementary product of two of its subgroups A and B, with one of them cyclic, is the natural first step toward such understanding. Among the first group theorists considering this specific case one should mention the seminal 1930's work of Neumann, followed by Ore, and later by Ito.

A skew morphism of a group is a generalization of the concept of an automorphism, which arose from the study of regular Cayley maps, but occurs more generally in the context of a group G expressible as a product AB of subgroups A and B with B cyclic and $A \cap B = \{1\}$. Specifically, a *skew morphism* of a group G is a bijection $\varphi\colon G \to G$ fixing the identity element of G and having the property that $\varphi(xy) = \varphi(x)\varphi^{\pi(x)}(y)$ for all $x, y \in G$, where $\pi(x)$ depends only on x (and is called the *power function* of φ). The *kernel* of φ is the subgroup of all $x \in A$ for which $\pi(x) = 1$.

Despite their original appearance in combinatorics, skew morphisms and their power functions possess a surprising number of algebraic properties. In our presentation, we present a number of such properties. Among them, we show that if G is any finite group, then the order of every skew morphism of G is less than $|G|$ (with an analogous result for automorphisms proved by Khoroshevski in the 1960's), and that the kernel of every skew morphism of a non-trivial finite group is non-trivial.

We also include a number of theorems about skew morphisms of finite abelian groups. For example, we determine all skew morphisms of the finite abelian groups whose order is prime, or the square of a prime, or the product of two distinct primes. We completely determine the finite abelian groups for which every skew morphism is an automorphism; these are precisely the cyclic groups C_n with $n = 4$ or $\gcd(n, \phi(n)) = 1$, and the elementary abelian 2-groups $C_2 \times \ldots \times C_2$. Finally, we survey some of the most recent classification results concerning skew morphisms of cyclic and dihedral groups.