Logical calculus for compact Hausdorff spaces via Boolean algebras with binary relations

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Y. Venema

Part 1: Dualities

Stone duality

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Theorem (Stone duality).
Stone is dually equivalent to BA.

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If B is a Boolean algebra, then the space X_B of ultrafilters of B with the topology given by $\{\alpha(a) : a \in B\}$, where

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Theorem (Stone representation)
Every Boolean algebra B is isomorphic to $\mathsf{Clop}(X_B)$.

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Our approach is based on the duality used in modal logic.

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1. $\Diamond 0 = 0$,
2. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.

Continuous relations

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2. $U \in \mathsf{Clop}(X) \Rightarrow R^{-1}[U] \in \mathsf{Clop}(X)$, where

$$R^{-1}[U] = \{x \in X : R[x] \cap U \neq \emptyset\}.$$

In other words, $R^{-1} : \mathsf{Clop}(X) \to \mathsf{Clop}(X)$ is well defined.

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Theorem (Esakia, 1974) R is continuous iff $\rho : X \to VX$ defined by $\rho(x) = R[x]$ is a well-defined continuous map, where VX is the Vietoris space.

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Theorem (Jónsson-Tarski representation)
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However, continuous relations satisfy the following symmetric condition.

For each closed set F both $R[F]$ and $R^{-1}[F]$ are closed.

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Therefore it is natural to study closed relations.

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What axioms does this binary relation validate?

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(S1) $0 \prec a \prec 1$ for each $a \in B$;

(S2) $a \prec b, c$ implies $a \prec b \wedge c$;

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$(\mathsf{Clop}(X), \prec)$ is a Boolean algebra with a subordination.

Quasi-Modal operators and pre-contact relations

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For $S \subseteq B$, let

$$\twoheaduparrow S = \{b \in B : \exists a \in S \text{ with } a \prec b\}$$

$$\twoheaddownarrow S = \{b \in B : \exists a \in S \text{ with } b \prec a\}.$$

Then

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Subordinations are in 1-1 correspondence with Düntch and Vakarelov's pre-contact relations.

$$a\delta b \;\text{ iff }\; a \not\prec \neg b.$$

Boolean algebras with subordinations

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Theorem (Celani, 2001, Dimov and Vakarelov, 2006) Every Boolean algebra with a subordination $(B, \prec)$ is isomorphic to $(\mathsf{Clop}(X), \prec)$ for some Stone space with a closed relation.

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Theorem (Celani, 2001, Dimov and Vakarelov, 2006) Every Boolean algebra with a subordination $(B, \prec)$ is isomorphic to $(\mathsf{Clop}(X), \prec)$ for some Stone space with a closed relation.

This correspondence can be extended to dualities of appropriate categories (G.B., N.B, S.S., Y.V., 2014).

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Every Sahlqvist modal formula has a first order-correspondent on relational structures.

1. R is reflexive iff $\Box p \to p$ is valid.
2. R is symmetric iff $p \to \Box\Diamond p$ is valid.
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(S5) $a \prec b$ implies $a \leq b$;

(S6) $a \prec b$ implies $\neg b \prec \neg a$;

(S7) $a \prec b$ implies there is $c \in B$ with $a \prec c \prec b$;

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Sahlqvist correspondence for similar languages were studied by (Balbiani and Kikot, 2012) and (Santoli, 2016).

Gleason cover

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Definition. An onto continuous map $\pi : X \to Y$ between compact Hausdorff spaces is called irreducible if the image of a proper closed set is proper.

The Gleason cover of a compact Hausdorff space Y is a pair (X, π), where X is an extremally disconnected (ED) Stone space and $\pi : X \to Y$ is an irreducible map.

Regular open sets

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Let $\mathcal{RO}(X)$ be the Boolean algebra of regular open subsets of X, where

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- $\bigwedge_{i \in I} U_i = \mathbf{Int} \bigcap_{i \in I} U_i$,
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Then R is irreducible iff $(B, \prec)$ satisfies (S8).

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Moreover, since X is also ED, $\mathsf{Clop}(X)$ is complete.

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A de Vries algebra is a pair $(B, \prec)$, where B is a complete Boolean algebra and $\prec$ is a compingent relation.

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Theorem (G.B, N.B. Sourabh, Venema, 2014) Every de Vries algebra is isomorphic to $(\mathsf{Clop}(X), \prec)$ for some Gleason space (X, R).

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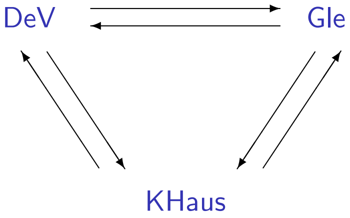
Corollary (de Vries, 1962) The category KHaus of compact Haudorff spaces is dual to the category DeV of de Vries algebras.



Part 2: Logical calculi

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We will consider formulas in the following language:

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A two sorted language to reason about de Vries algebras was investigated by Balbiani, Tinchev and Vakarelov (2007).

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(I2)-(I3) imply (S4).

Axiomatization

Other axioms can be rewritten as follows.

(I4) $a \rightsquigarrow b \leq a \to b$;

(I5) $a \rightsquigarrow b = \neg b \rightsquigarrow \neg a$;

(I6) $a \rightsquigarrow b = 1$ implies $\exists c : a \rightsquigarrow c = 1$ and $c \rightsquigarrow b = 1$;

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Simple (I1)-(I5)-algebras correspond to contact algebras.

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These are $\forall\exists$-statements.

Non-standard rules

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Theorem. (Chang-Łos-Suszko) A class of structures is axiomatized by $\forall\exists$-statements iff it is an inductive class.

Hierarchy

Formulas φ	$\leftrightsquigarrow$	varieties
Rules Γ/φ	$\leftrightsquigarrow$	quasi-varieties
Rules Γ/Δ	$\leftrightsquigarrow$	universal classes
?	$\leftrightsquigarrow$	inductive classes

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A non-standard rule is one of the form:

$$(\rho) \quad \frac{F(\bar{\varphi}, \bar{p}) \to \chi}{G(\bar{\varphi}) \to \chi}$$

where χ is a formula variable, and F, G are formulas, each involving formula variables $\bar{\varphi}$, and with F involving a fresh tuple $\bar{p}$ of proposition letters.

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With the rule (ρ), we associate the first-order formula Φ_ρ, defined as:

$$\Phi_\rho \;:=\quad \forall \bar{a}, b \in B \left(G(\bar{a}) \nleq b \;\Rightarrow\; \exists \bar{c} : \; F(\bar{a}, \bar{c}) \nleq b \right)$$

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Hierarchy

Logics	$\leftrightsquigarrow$	varieties
Consequence relations	$\leftrightsquigarrow$	quasi-varieties
Multi consequence relations	$\leftrightsquigarrow$	universal classes
Non-standard rule calculi	$\leftrightsquigarrow$	inductive classes

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$(\rho 6)$ corresponds to $(I6)$
$(\rho 7)$ corresponds to $(I7)$

Completeness

Theorem. (G. B., N. B., Santoli, Venema, 2017)
Let L be obtained by adding non-standard rules $\{\rho_i\}_{i\in I}$ to (I1)-(I5). Then L is sound and complete wrt the class of algebras satisfying $\{\Phi_{\rho_i}\}_{i\in I}$.

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What about topological completeness?

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Theorem.

- Compingent algebras are closed under MacNeille completions.

Completeness

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Corollary (G. B., N. B., Santoli, Venema, 2017)

1. (I1)-(I5) + $(\rho 6), (\rho 7)$ is sound and complete wrt de Vries algebras.
2. (I1)-(I5) + $(\rho 6), (\rho 7)$ is sound and complete wrt Gleason spaces.
3. (I1)-(I5) + $(\rho 6), (\rho 7)$ is sound and complete wrt compact Haudorff spaces.

Thank you!