Canonical extensions of lattices are more than perfect

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Andrew and Maria

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(i) **in distributive case**: Priestley duality as a natural duality,

(ii) **in non-distributive case** (with **Andrew** and **Maria**): a topological representation due to M. Ploščica (1995) which presents **the classical one due to A. Urquhart (1978)** in the spirit of the natural dualities.

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Let $\mathbf{B}$ be a Boolean algebra (with operators) and let $X_{\mathbf{B}}$ be the Stone space dual to $\mathbf{B}$, i.e., $X_{\mathbf{B}}$ is the set of ultrafilters of $\mathbf{B}$ with an appropriate topology. (Stone duality tells us that we may identify the Boolean algebra $\mathbf{B}$ with the Boolean algebra of *clopen* subsets of the Stone space $X_{\mathbf{B}}$.)

The canonical extension $\mathbf{B}^{\delta}$ of $\mathbf{B}$ is the Boolean algebra $\wp(X_{\mathbf{B}})$ of *all* subsets of the set $X_{\mathbf{B}}$ of ultrafilters of $\mathbf{B}$ (with the operators extended in a natural way).

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Thus, roughly speaking, Jónsson and Tarski obtained $\mathbf{B}^{\delta}$ from the Stone space $X_{\mathbf{B}}$ by forgetting the topology.

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- An equational class of algebras is said to be canonical if it is closed under the formation of canonical extensions.
- When the members of the class of lattice-based algebras are the algebraic models of a logic, canonicity leads to completeness results for the associated logic.
- That is partly why the canonical extensions have been important and have been of an interest to logicians, too.

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The canonical extension $\mathbf{L}^{\delta}$ of $\mathbf{L}$ is the **doubly algebraic distributive** lattice $\mathrm{Up}(X_{\mathbf{L}})$ of *all* up-sets of the ordered set $\langle X_{\mathbf{L}}; \subseteq \rangle$ of prime filters of $\mathbf{L}$.

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Thus, again, $\mathbf{L}^{\delta}$ is obtained from the Priestley space $X_{\mathbf{L}}$ by forgetting the topology.

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- **C** is called a compact completion of **L** if, for any subsets $A, B \subseteq L$ we have $\bigwedge A \leqslant \bigvee B$ implies the existence of finite subsets $A' \subseteq A$, $B' \subseteq B$ with $\bigwedge A' \leqslant \bigvee B'$.
  (**Equivalently**: if for every filter F and every ideal J of **L**, we have $\bigwedge F \leqslant \bigvee J$ implies $F \cap J \neq \varnothing$.)

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- Abstractly, **the canonical extension of a BL L** has been defined as a **dense** and **compact completion** of **L.**
- Concretely, they constructed $\mathbf{L}^\delta$ as the complete lattice of **Galois-stable sets of the polarity** R between the filter lattice Filt(**L**) and the ideal lattice Idl(**L**) of **L**:

$$(F, I) \in R :\iff F \cap I \neq \emptyset.$$

Duals of lattice-based algebras via natural dualities

- For the class $\mathcal{B}$ of BAs and $\mathbf{B} \in \mathcal{B}$, we define the dual of $\mathbf{B}$ to be $D(\mathbf{B}) = \mathcal{B}(\mathbf{B}, \underline{\mathbf{2}})$, the set of all homomorphisms from $\mathbf{B}$ to $\underline{\mathbf{2}}$. There is a one-to-one correspondence between $\mathcal{B}(\mathbf{B}, \underline{\mathbf{2}})$ and the set of ultrafilters of $\mathbf{B}$.

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- For the class $\mathcal{D}$ of BDLs and $\mathbf{D} \in \mathcal{D}$, we define $D(\mathbf{D}) = \mathcal{D}(\mathbf{D}, \underline{\mathbf{2}})$. There is a one-to-one correspondence between $\mathcal{D}(\mathbf{D}, \underline{\mathbf{2}})$ and the prime filters of $\mathbf{D}$.

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Ploščica's representation for bounded lattices

- Let $\mathbf{L}$ be a bounded lattice. Ploščica's dual of $\mathbf{L}$ is $D(\mathbf{L}) = \mathbf{X}_{\mathbf{L}} := (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R, \mathcal{T})$ where binary (reflexive) relation R for $f, g \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ is defined as follows: $(f, g) \in R$ *if* $\forall\, a \in \mathit{dom}\, f \cap \mathit{dom}\, g,\ f(a) \leqslant g(a)$.
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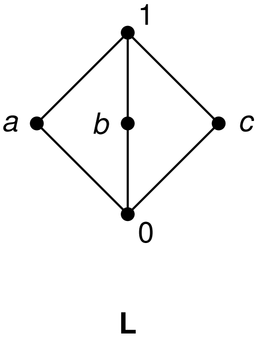
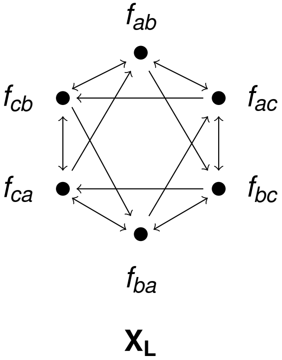
Theorem (Ploščica, 1995)

Let $\mathbf{L} \in \mathcal{L}$. Then $\mathbf{L} \cong ED(\mathbf{L})$ via the map $a \mapsto e_a$ where $e_a\colon (\mathbf{X_L}, \mathcal{T}) \to \underset{\sim}{\mathbf{2}}_{\mathcal{T}}$ is defined by $e_a(f) = f(a)$.

Example of the dual graph of a bounded lattice





The modular lattice $\mathbf{L} = \mathbf{M}_3$ and its graph $\mathbf{X_L} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R)$.

We define $f_{xy} \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ by $f_{xy}^{-1}(1) = {\uparrow}x$ and $f_{xy}^{-1}(0) = {\downarrow}y$.

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$$\begin{aligned} \mathrm{D}\colon \mathbf{L} &\longmapsto \mathbf{X}_{\mathcal{T}} := (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R, \mathcal{T}), \\ {}^{\flat}\colon \mathbf{X}_{\mathcal{T}} &\longmapsto \mathbf{X} := (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R), \\ \mathrm{C}\colon \mathbf{X} &\longmapsto \mathrm{C}(\mathbf{X}) := \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}}). \end{aligned}$$

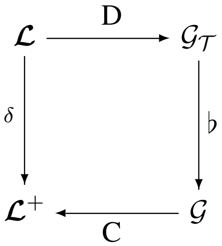

Figure: Factorisation of $^{\delta}$ in $\mathcal{L}$ (**Craig, Haviar, Priestley [ACS, 2013]**)

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Theorem

Let $\mathbf{L} \in \mathcal{L}$ *be a BL. Let* $\mathrm{D}^\flat(\mathbf{L}) = \mathbf{X} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R)$ *be Ploščica's dual of* $\mathbf{L}$ *(without topology). The lattice* $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ *ordered by*

$$\varphi \leqslant \psi :\Longleftrightarrow \varphi^{-1}(1) \subseteq \psi^{-1}(1)$$

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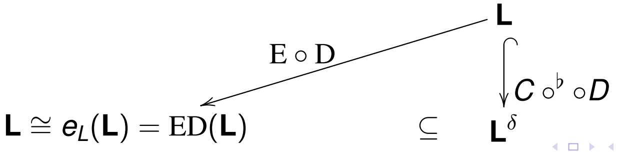
We proved the density and compactness of $\mathrm{C}(\mathbf{X})$ (technical).



Dual representation: perfect lattices vs RS frames (Dunne, Gehrke,Palmigiano [2005], Gehrke [2006])

C is a perfect lattice if **C** is complete, and for all $c \in C$,

$$c = \bigvee\{\, j \in J^{\infty}(\mathbf{C}) \mid j \leqslant c \,\} = \bigwedge\{\, m \in M^{\infty}(\mathbf{C}) \mid c \leqslant m \,\}.$$

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From RS frames to perfect lattices: Let $\mathbb{F} = (X, Y, R)$ be an **RS frame.** A Galois connection between $\wp(X)$ and $\wp(Y)$ is defined as follows for $A \subseteq X$, $B \subseteq Y$:

$$R_{\triangleright}(A) = \{\, y \in Y \mid \forall\, a \in A, aRy \,\}\ R_{\triangleleft}(B) = \{\, x \in X \mid \forall b \in B, xRb \,\}.$$

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$$R_\rhd(A) = \{ y \in Y \mid \forall\, a \in A, aRy \} \; R_\lhd(B) = \{ x \in X \mid \forall b \in B, xRb \}.$$

Then $\mathcal{G}(\mathbb{F}) := \{ A \subseteq X \mid A = R_\lhd \circ R_\rhd(A) \}$ is a **perfect lattice.**

Frames associated to bounded lattices

For a frame $\mathbb{F} = (X, Y, R)$ and $x \in X$ and $y \in Y$ we define

$$xR := \{\, y \in Y \mid xRy \,\} \quad \text{and} \quad Ry := \{\, x \in X \mid xRy \,\}.$$

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(S) for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$,
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- (i) for every $x \in X$ there exists $y \in Y$ such that $\neg(xRy)$ and $\forall w \in X\ ((w \neq x \ \ \& \ \ xR \subseteq wR) \Rightarrow wRy)$;
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Defining RS graphs

Lemma

Let $\mathbf{L}$ *be a BL. Then* $\mathbf{X_L} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R)$ *satisfies:*

(S) *for every* $f, g \in X$*, if* $f \neq g$ *then* $f^{-1}(1) \neq g^{-1}(1)$ *or* $f^{-1}(0) \neq g^{-1}(0)$*;*

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Observe the following for $f, g, h \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$:

$$f^{-1}(1) \subseteq h^{-1}(1) \text{ iff } hR \subseteq fR \text{ and } g^{-1}(0) \subseteq h^{-1}(0) \text{ iff } Rh \subseteq Rg.$$

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Hence we may rewrite the conditions (S) and (R) above, and define them for any reflexive graph $\mathbf{X} = (X, R)$, as follows:

(S) for every $x, y \in X$, if $x \neq y$ then $xR \neq yR$ or $Rx \neq Ry$;

(R) (i) for all $x, z \in X$, if $zR \subsetneq xR$ then $(z, x) \notin R$;
(ii) for all $y, z \in X$, if $Rz \subsetneq Ry$ then $(y, z) \notin R$.

The (Ti) property for graphs and frames

Let $\mathbf{X} = (X, R)$ be a reflexive graph and consider the property:

(Ti) for all $x, y \in X$, if $(x, y) \in R$, then there exists $z \in X$ such that $zR \subseteq xR$ and $Rz \subseteq Ry$.

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Let $\mathbb{F} = (X_1, X_2, R)$ be a frame. The (Ti) condition for frames is motivated by the (Ti) condition on graphs:

(Ti) for every $x \in X_1$ and for every $y \in X_2$, if $\neg(xRy)$ then there exist $w \in X_1$ and $z \in X_2$ such that

(i) $\neg(wRz)$;
(ii) $xR \subseteq wR$ and $Ry \subseteq Rz$;
(iii) for every $u \in X_1$, if $u \neq w$ and $wR \subseteq uR$ then uRz;
(iv) for every $v \in X_2$, if $v \neq z$ and $Rz \subseteq Rv$ then wRv.

The (Ti) property for frames in a special case

If the frame $\mathbb{F} = (X_1, X_2, R)$ is $\mathbb{F} = (X, X, \leqslant)$, then (Ti) says that for all $x, y \in X$, if $x \nleqslant y$ then there are $w, z \in X$ such that

(i) $w \nleqslant z$;
(ii) $w \leqslant x$ and $y \leqslant z$;
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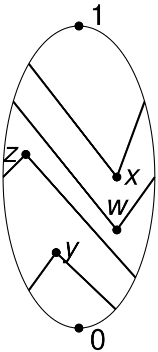

TiRS graphs and frames

Definition

A TiRS graph (frame) is a graph (frame) that satisfies the conditions (R), (S) and (Ti), i.e., it is an RS graph (frame) that satisfies the condition (Ti).

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Proposition

For any bounded lattice $\mathbf{L}$,

(i) *its Ploščica's dual* $\mathrm{D}^\flat(\mathbf{L}) = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R)$ *is a TiRS graph;*

(ii) *the frame* $\mathbb{F}(\mathbf{L}) = (\mathrm{Filt}_M(\mathbf{L}), \mathrm{Idl}_M(\mathbf{L}), R)$ *is a TiRS frame.*

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Theorem

There is a one-to-one correspondence between TiRS graphs and TiRS frames.

Proof: from graphs to frames

Given a graph $\mathbf{X} = (X, R)$, define $\sim_1$ and $\sim_2$ for $x, y \in X$ by

$$x \sim_1 y \quad \text{if} \quad xR = yR \qquad x \sim_2 y \quad \text{if} \quad Rx = Ry.$$

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For a graph $\mathbf{X} = (X, R)$, the assoc. frame $\rho(\mathbf{X})$ is defined by:

$$\rho(\mathbf{X}) = (X/{\sim_1}, X/{\sim_2}, R_{\rho(\mathbf{X})}) \text{ where } [x]_1 R_{\rho(\mathbf{X})} [y]_2 \iff (x, y) \notin R.$$

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Proof-cont.: from frames to graphs

Definition

Let $\mathbf{F} = (X_1, X_2, R)$ *be a frame. The* *associated graph* $\mathrm{gr}(\mathbf{F})$ *is* $(H_\mathbf{F}, K_\mathbf{F})$ *where the vertex set* $H_\mathbf{F}$ *is the subset of* $X_1 \times X_2$ *of all pairs* (x, y) *that satisfy the following conditions:*

(a) $\neg(xRy)$,

(b) *for every* $u \in X_1$, *if* $u \neq x$ *and* $xR \subseteq uR$ *then* uRy,

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Proposition

Let $\mathbf{F} = (X_1, X_2, R)$ *be a TiRS frame. The assoc. graph* $\rho(\mathbf{F})$ *is a TiRS graph.*

TiRS frames and TiRS graphs: correspondence

- Two graphs $\mathbf{X} = (X, R_X)$ and $\mathbb{Y} = (Y, R_Y)$ are isomorphic if there exists a bijective map $\alpha\colon X \to Y$ such that

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- Two frames $\mathbf{F} = (X_1, X_2, R_F)$ and $\mathbb{G} = (Y_1, Y_2, R_G)$ are isomorphic if there exists a pair (β_1, β_2) of bijective maps $\beta_i\colon X_i \to Y_i$ $(i = 1, 2)$ with

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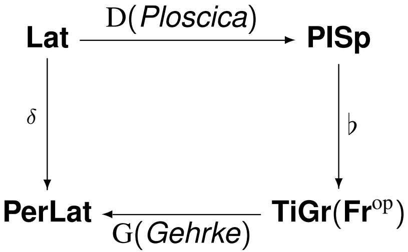
Theorem

Let $\mathbf{X} = (X, R_X)$ be a TiRS graph and $\mathbf{F} = (X_1, X_2, R)$ be a TiRS frame. Then

(a) *the graphs $\mathbf{X}$ and $\mathrm{gr}(\rho(\mathbf{X}))$ are isomorphic;*

(b) *the frames $\mathbf{F}$ and $\rho(\mathrm{gr}(\mathbf{F}))$ are isomorphic.*

CEs of BLs (Craig, Gouveia, Haviar [AU, 2015])



Theorem

Let **L** *be a bounded lattice and* $\mathbf{X} = \mathrm{D}^{\flat}(\mathbf{L})$ *be its dual (Ploščica's) TiRS graph. Let* $\rho(\mathbf{X})$ *be the TiRS frame associated to* **X** *and* $\mathrm{G}(\rho(\mathbf{X}))$ *be its corresponding (Gehrke's) perfect lattice of Galois-closed sets.*
Then the lattice $\mathrm{G}(\rho(\mathbf{X}))$ *is the canonical extension of* **L**.

Question

Is every TiRS graph $\mathbf{X} = (X, R)$ of the form $\mathrm{D}^{\flat}(\mathbf{L}) = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R)$ *for some bounded lattice* $\mathbf{L}$*?*

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Answer

No. Every poset is a TiRS graph. A poset is said to be **representable** *if it is the underlying poset of some Priestley space and hence the untopologized dual of some bounded distributive lattice. It is known that* **non-representable posets exist** *and hence non-representable TiRS graphs exist.*

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Problem 1

Which TiRS graphs arise as duals of bounded lattices?

We proved that the RS frame associated to the CE is always TiRS. Hence, by the correspondence between RS frames and perfect lattices, one could ask whether also conversely:

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Problem 2

Consider the perfect lattice which corresponds to a TiRS frame. In addition to being perfect, what additional properties of the complete lattice arise as a result of it coming from an RS frame which also satisfies (Ti)?

(PTi) condition

A perfect lattice is (PTi) if for all $x \in J^\infty(\mathbf{C})$ and $y \in M^\infty(\mathbf{C})$, if $x \nleqslant y$ then there exist $w \in J^\infty(\mathbf{C})$, $z \in M^\infty(\mathbf{C})$ such that

(i) $w \leqslant x$ and $y \leqslant z$

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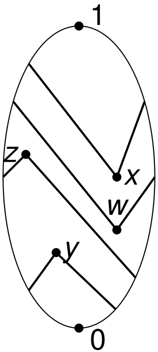

Perfect lattices dual to TiRS structures are (PTi)

Proposition

Let **C** *be a* **perfect lattice***. If* **C** *satisfies* (PTi) *then the assoc.* **RS-frame F(C)** $= (J^{\infty}(\mathbf{C}), M^{\infty}(\mathbf{C}), \leqslant)$ *satisfies* (Ti)*, and so it is a TiRS frame.*

Theorem

Let $\mathbb{F} = (X, Y, R)$ *be a TiRS frame. Then the* **perfect lattice** $\mathcal{G}(\mathbb{F})$ **of the Galois closed sets** *satisfies* (PTi)*.*

Corollary

The canonical extension of a bounded lattice is more than perfect - it is a perfect lattice that satisfies (PTi).

Proofs (key ingredients)

Theorem

Let $\mathbb{F} = (X, Y, R)$ be a TiRS frame. Then the **perfect lattice $\mathcal{G}(\mathbb{F})$ of the Galois closed sets** *satisfies (PTi).*

- By **Gehrke** the completely join-irreducible elements and completely meet-irreducible elements of $\mathcal{G}(\mathbb{F})$ are:

$$J^{\infty}(\mathcal{G}(\mathbb{F})) = \{\, (R_{\triangleleft} \circ R_{\triangleright})(\{x\}) \mid x \in X \,\},$$
$$M^{\infty}(\mathcal{G}(\mathbb{F})) = \{\, Ry \mid y \in Y \,\}.$$

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Proofs (key ingredients) - cont.

Corollary

The canonical extension of a bounded lattice is more than perfect - it is a perfect lattice that satisfies (PTi).

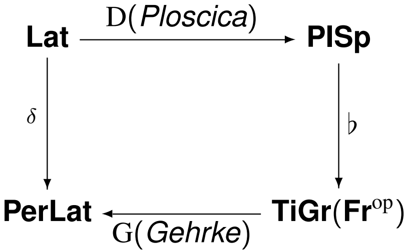
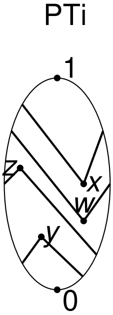
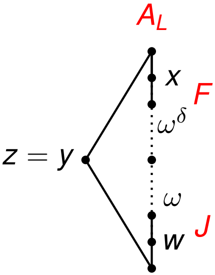


Figure: CEs for BLs (Craig, Gouveia, Haviar)

The (PTi) lattice A_L that is not a CE [Sept 2017]

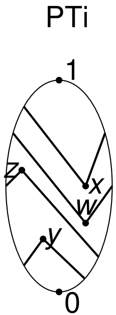


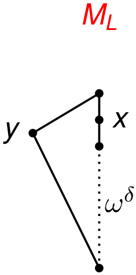


A_L: a PTi lattice that is not a CE

(Compactness of a CE: for every filter F and every ideal J, $\bigwedge F \leqslant \bigvee J$ implies $F \cap J \neq \varnothing$.)

The perfect but not (PTi) lattice M_L [Sept 2017]





M_L: a perfect but non-PTi lattice

TiRS graph and TiRS frame morphisms

Definition

Let $\mathbf{X} = (X, R_X)$ and $\mathbf{Y} = (Y, R_Y)$ be TiRS graphs. A TiRS graph morphism is a map $\varphi\colon X \to Y$ that satisfies the following conditions:

(i) for $x_1, x_2 \in X$, if $(x_1, x_2) \in R_X$ then $(\varphi(x_1), \varphi(x_2)) \in R_Y$;

(ii) for $x_1, x_2 \in X$, if $x_1 R_X \subseteq x_2 R_X$ then $\varphi(x_1) R_Y \subseteq \varphi(x_2) R_Y$;

(iii) for $x_1, x_2 \in X$, if $R_X x_1 \subseteq R_X x_2$ then $R_Y \varphi(x_1) \subseteq R_Y \varphi(x_2)$.

Every graph isomorphism and its inverse are **TiRS graph morphisms.**

TiRS graph and TiRS frame morphisms - cont.

Definition

Let $\mathbf{F} = (X_1, X_2, R_F)$ and $\mathbf{G} = (Y_1, Y_2, R_G)$ be TiRS frames. A TiRS frame morphism $\psi\colon \mathbf{F} \to \mathbf{G}$ is a pair $\psi = (\psi_1, \psi_2)$ of maps $\psi_1 : X_1 \to Y_1$ and $\psi_2 : X_2 \to Y_2$ that satisfies the following conditions:

(i) for $x \in X_1$ and $y \in X_2$, if $\psi_1(x) R_G \psi_2(y)$ then $x R_F y$;

(ii) for $x, w \in X_1$, if $xR_F \subseteq wR_F$ then $\psi_1(x)R_G \subseteq \psi_1(w)R_G$;

(iii) for $y, z \in X_2$, if $R_F y \subseteq R_F z$ then $R_G \psi_2(y) \subseteq R_G \psi_2(z)$;

(iv) for $x \in X_1$ and $y \in X_2$, if $(x, y) \in H_{\mathbf{F}}$ then $(\psi_1(x), \psi_2(y)) \in H_{\mathbf{G}}$.

Every frame isomorphism is a **TiRS frame morphism.**

TiRS graphs and frames: a full categorical framework

Theorem

Let $\mathbf{X} = (X, R_X)$ *and* $\mathbf{Y} = (Y, R_Y)$ *be TiRS graphs and let* $\mathbf{F} = (X_1, X_2, R_F)$ *and* $\mathbf{G} = (Y_1, Y_2, R_G)$ *be TiRS frames.*

(1) *If* $\varphi\colon \mathbf{X} \to \mathbf{Y}$ *is a* **TiRS graph morphism** *and for* $\rho(\varphi)_1\colon X/{\sim_1} \to Y/{\sim_1}$ *and* $\rho(\varphi)_2\colon X/{\sim_2} \to Y/{\sim_2}$ *are the maps defined for all* $x \in X$ *by* $\rho(\varphi)_1([x]_1) = [\varphi(x)]_1$ *and* $\rho(\varphi)_2([x]_2) = [\varphi(x)]_2$*, then the pair* $\rho(\varphi) = (\rho(\varphi)_1, \rho(\varphi)_2)$ *is a TiRS frame morphism from* $\rho(\mathbf{X})$ *to* $\rho(\mathbf{Y})$*.*

(2) *If the pair* $\psi = (\psi_1, \psi_2)\colon \mathbf{F} \to \mathbf{G}$ *is a* **TiRS frame morphism***, then the map* $\mathrm{gr}(\psi)\colon \mathrm{gr}(\mathbf{F}) \to \mathrm{gr}(\mathbf{G})$ *defined for* $(x, y) \in H_F$ *by* $\mathrm{gr}(\psi)(x, y) = (\psi_1(x), \psi_2(y))$ *is a TiRS graph morphism.*

A full categorical framework - cont.

Theorem

Let $\mathbf{X} = (X, R_X)$ *and* $\mathbf{Y} = (Y, R_Y)$ *be TiRS graphs and let* $\mathbf{F} = (X_1, X_2, R_F)$ *and* $\mathbf{G} = (Y_1, Y_2, R_G)$ *be TiRS frames.*

(3) *If* $\varphi\colon \mathbf{X} \to \mathbf{Y}$ *is a TiRS graph morphism, then* $\mathrm{gr}(\rho(\varphi)) \circ \alpha_X = \alpha_Y \circ \varphi$.

(4) *If* $\psi\colon \mathbf{F} \to \mathbf{G}$ *is a TiRS frame morphism, then* $\rho(\mathrm{gr}(\psi)) \circ \beta_F = \beta_G \circ \psi$.

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\quad\varphi\quad} & \mathbf{Y} \\ \Big\downarrow{\scriptstyle \alpha_X} & & \Big\downarrow{\scriptstyle \alpha_Y} \\ \mathrm{gr}(\rho(\mathbf{X})) & \xrightarrow{\mathrm{gr}(\rho(\varphi))} & \mathrm{gr}(\rho(\mathbf{Y})) \end{array} \qquad \begin{array}{ccc} \mathbf{F} & \xrightarrow{\quad\psi\quad} & \mathbf{G} \\ \Big\downarrow{\scriptstyle \beta_F} & & \Big\downarrow{\scriptstyle \beta_G} \\ \rho(\mathrm{gr}(\mathbf{F})) & \xrightarrow{\rho(\mathrm{gr}(\psi))} & \rho(\mathrm{gr}(\mathbf{G})) \end{array}$$

Proposition

Every **finite RS frame is** *a* **TiRS frame**.

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The following result generalises Birkhoff's dual representation of finite distributive lattices via finite posets.

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There exists a dual representation of arbitrary finite lattices via finite TiRS graphs.

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Proof.

Finite lattices = Finite perfect lattices

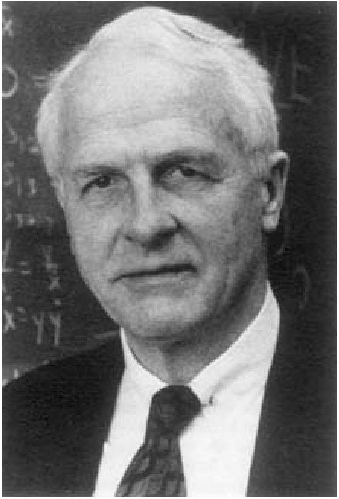
$\updownarrow$

Finite RS frames = Finite TiRS frames

$\updownarrow$

Finite TiRS graphs

Garrett Birkhoff (1911 – 1996)



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We wonder whether this representation could bring a new light to the famous **Finite Congruence Lattice Problem (FCLP)** which has been open for decades:

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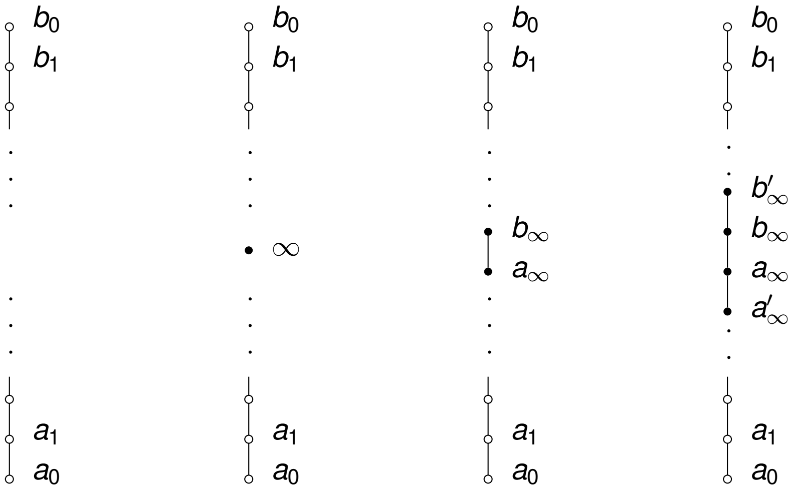
We wonder whether this representation could bring a new light to the famous **Finite Congruence Lattice Problem (FCLP)** which has been open for decades:

Problem 3

Is every finite lattice isomorphic to the congruence lattice of some finite algebra?

The DM completion vs the CE



$\mathbf{L} = \omega \oplus \omega^\partial$ $\qquad$ $DM(\mathbf{L})$ $\qquad$ $\mathbf{L}^\delta$ $\qquad$ $(\mathbf{L}^\delta)^\delta$