# Interpretability lattice of clonoids

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# Disclaimer

- The PCSP complexity results are not ours
- Pioneers of PCSP: Per Austrin, Joshua Brakensiek, Venkatesan Guruswami, and Johan Håstad
- Any errors, typos etc. in the presentation belong to A. Kazda

# Promise Constraint Satisfaction

- $\mathbb{A}, \mathbb{B}$ are relational structures, $\mathbb{A} \to \mathbb{B}$
- $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$: Input relational structure $\mathbb{C}$
  - Output "Yes" if $\mathbb{C} \to \mathbb{A}$
  - Output "No" if $\mathbb{C} \not\to \mathbb{B}$
- Example: $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_4)$.

# $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$

- $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ are all polymorphisms from $\mathbb{A}$ to $\mathbb{B}$
- Polymorphism $f \colon \mathbb{A}^n \to \mathbb{B}$ sends $R^{\mathbb{A}}$ into $R^{\mathbb{B}}$
- $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ is great for classifying complexity of $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$
- Can't compose, but can take minors:

$$f(x_1, x_2, x_3, x_4, x_5) \in \mathrm{Pol}(\mathbb{A}, \mathbb{B}) \Rightarrow f(x_{\mathbf{2}}, x_{\mathbf{2}}, x_{\mathbf{16}}, x_{\mathbf{4}}, x_{\mathbf{5}}) \in \mathrm{Pol}(\mathbb{A}, \mathbb{B})$$

# Clonoids (AKA minor closed sets)

- A functional clonoid $\mathcal{C}$ on sets $A, B$ is a nonempty family of operations from $A$ to $B$ closed under taking minors
- Taking minors: $\sigma\colon [n] \to [m]$ sends $n$-ary $f$ to $m$-ary $f^\sigma$ where

$$f^\sigma(x_1, \ldots, x_m) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

- Each $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ is a clonoid
- If we just care about minors and identities, we get abstract clonoids (="clonoids")
- Libor Barto: Abstract clonoids = finitary Set-endofunctors

# Clonoid homomorphisms

- $\phi\colon \mathcal{C} \to \mathcal{D}$ preserves arity and commutes with taking minors
- Another view: homomorphism sends identities of $\mathcal{C}$ to identities of $\mathcal{D}$ ...can interpret $\mathcal{C}$ in $\mathcal{D}$
- Example:

$$f(x,x,y) \approx g(x,y,y,z) \Rightarrow \phi(f)(x,x,y) \approx \phi(g)(x,y,y,z)$$

- Jakub Opršal: For $\mathbb{A}, \mathbb{B}, \mathbb{A}', \mathbb{B}'$ finite relational structures $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) \to \mathrm{Pol}(\mathbb{A}', \mathbb{B}')$ gives a reduction from $\mathrm{PCSP}(\mathbb{A}', \mathbb{B}')$ to $\mathrm{PCSP}(\mathbb{A}, \mathbb{B})$

# Interpretability lattice

- "$\mathcal{C} \to \mathcal{D}$" is a quasiorder on clonoids – factorize to get the poset $\mathcal{L}$
- Libor Barto: $\mathcal{L}$ is alg-universal
- $\mathcal{L}$ is also a lattice
- Top element: Any clonoid with a constant operation $f(x) \approx f(y)$
- Bottom element: Essentially unary operations, only trivial identities

# Meet

- $\mathcal{C} \wedge \mathcal{D}$ has $n$-ary part $(f, g)$ where $f \in \mathcal{C}$ and $g \in \mathcal{D}$
- For functional clonoids and $f: A_1^n \to B_1$, $g: A_2^n \to B_2$ pick $(f, g): A_1^n \times A_2^n \to B_1 \times B_2$ defined componentwise
- Infinite meets are similar

# Join

- Abstract clonoids: Identities of $\mathcal{C} \vee \mathcal{D}$ are the disjoint union of identities of $\mathcal{C}$ and $\mathcal{D}$
- Functional clonoids: $\mathcal{C}$ goes from $A_1$ to $B_1$ and $\mathcal{D}$ goes from $A_2$ to $B_2$
- Then $\mathcal{C} \vee \mathcal{D}$ goes from $A_1 \cup A_2$ to $B_1 \cup B_2 \cup \{\star\}$
- Operations of $\mathcal{C} \vee \mathcal{D}$ originate from $\mathcal{C}$ or $\mathcal{D}$.
- Output $\star$ for "undefined" (handle with care)
- Infinite joins are similar

# An essential chain

- $\mathcal{O}_k \ldots$ essentially at most $k$-ary nonconstant operations
- Can realize $\mathcal{O}_k$ by idempotent operations on $\{0, 1\}$
- $\mathcal{O}_1 \to \mathcal{O}_2 \to \mathcal{O}_3 \to \ldots$
- PCSP($\mathcal{O}_i$) are all NP-hard (reduce from GapLabelCover)
- Let $\mathcal{O}_{<\infty} = \bigvee_{i=1}^{\infty} \mathcal{O}_i$

# The largest nontrivial clonoid class

- Let $\mathcal{O}_\infty$ be the clonoid of all idempotent operations on $\{0, 1\}$
- If $\mathcal{C}$ is a clonoid without a constant operation then $\mathcal{C} \to \mathcal{O}_\infty$
- $f \in \mathcal{C}$ goes to $f^O(x_1, \ldots, x_n)$ where eg.

$$f^O(111010) = \begin{cases} 0 & \mathcal{C} \models f(x_\mathbf{1} x_\mathbf{1} x_\mathbf{1} x_\mathbf{0} x_\mathbf{1} x_\mathbf{0}) \approx f(x_\mathbf{0} x_\mathbf{0} x_\mathbf{0} x_\mathbf{0} x_\mathbf{0} x_\mathbf{0}) \\ 1 & \text{else} \end{cases}$$

# $\mathcal{O}_{<\infty}$ vs. $\mathcal{O}_\infty$ clonoids

- Let $\mathcal{O}_{<\infty} = \bigvee_{i=1}^{\infty} \mathcal{O}_i$
- From definition of $\vee$, we can construct $\mathcal{O}_{<\infty}$ as a functional clonoid
- Let $\mathcal{O}_\infty$ be the class of all idempotent operations on $\{0, 1\}$
- $\mathcal{O}_{<\infty}$ is strictly below $\mathcal{O}_\infty$
- Reason: $\mathcal{O}_\infty$ has totally symmetric idempotent operations of all arities that are minors of each other, but $\mathcal{O}_{<\infty}$ does not

# No finite functional clonoids in $[\mathcal{O}_{<\infty}, \mathcal{O}_\infty)$

- A functional clonoid of operations $A^n \to B$ is finite if $A, B$ are finite
- If $\mathcal{C}$ is a clonoid above or equivalent to $\mathcal{O}_{<\infty}$, then $\mathcal{C}$ realizes any system of bounded arity linear identities true in $\mathcal{O}_\infty$
- $\mathcal{C}$ realizes any finitary part of $\mathcal{O}_\infty$ and for each $n$, there are only finitely many $n$-ary operations $A^n \to B$ in $\mathcal{C}$
- Compactness gives that $\mathcal{C}$ realizes all linear identities true in $\mathcal{O}_\infty$

# Open problems

- What interesting chains/cochains are there in $\mathcal{L}$?
- Where are the hard PCSPs? (Near the bottom of $\mathcal{L}$)
- Is there a weaker notion than clonoid homomorphism that captures PCSP?
- What does the lattice of finite functional clonoids look like?

Thank you for your attention.