Lattice valued structures

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Introduction: Boolean-valued models

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As known, to a consistent theory $\mathcal{T}$ in the classical first-order language, one can associate the *Lindenbaum algebra* $B(\mathcal{T})$. It is a Boolean algebra of equivalence classes of formulas in $\mathcal{T}$ under the logical equivalence. It can be shown that any Boolean algebra is isomorphic to $B(\mathcal{T})$ for a suitable theory $\mathcal{T}$.

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Similarly, for the consistent theory in the intuitionistic first-order logic, the corresponding algebraic structure of equivalent classes is a Heyting algebra.

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In a similar way, complete Heyting algebras are related to intuitionistic first-order logic.

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Instead of characteristic functions, new objects became all functions from sets to B.
In this way the universe of *Boolean-valued sets* denoted by $V^{(B)}$ was obtained, consisting of much more functions than there were sets previously.

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(vii) $[\![u = v]\!] \wedge [\![\phi(u)]\!] \leq [\![\phi(v)]\!]$ *for any formula* ϕ.

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The mapping μ is a **fuzzy set on A**, or a **fuzzy subset of A** and these lattices are structures of **truth or membership values**.

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Structures suitable to fulfill these requirements are **residuated lattices** and related ordered structures.

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$\mu_p := \{x \in X \mid \mu(x) \geqslant p\}$ - p-**cut**, a **cut set**, **cut** of μ.

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For a nullary operation (constant) $c \in F, \quad \mu(c) = 1$.

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Theorem

Let $\mathcal{A}$ be an algebra. Then $\mu : A \to L$ is a fuzzy subalgebra of $\mathcal{A}$ if and only if for every $p \in L$, the cut set μ_p is a classical subalgebra of $\mathcal{A}$.

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Let $[0, 1]$ be the unit real interval. Then $([0, 1], \wedge, \vee)$ is a complete (distributive) lattice under the usual ordering $\leqslant$ and with
$x \wedge y = \min(x, y); \quad x \vee y = \max(x, y)$.

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  Boolean algebras, finite distributive lattices, chains, are examples of Heyting algebras.

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In the fuzzy framework, it is reasonable to introduce a **set with L-equality** (**set with fuzzy equality**) as a pair $(X, \mathbb{E})$ where $X \neq \emptyset$ and $\mathbb{E}$ is a fuzzy equality, i.e., a fuzzy equivalence satisfying a separation property.

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Remark

It would be more precise to define an L-algebra as a triple $(A, \mathbb{E}^A, F^A)$, so that the classical equality "$=$" is formally excluded (as it is done in original paper by Bělohlávek and Vychodil). Still, classical equality is not excluded in the definition of a set with a fuzzy equality. The present definition does not create confusions and it is convenient to have an algebra as a skeleton.

Let $\overline{\mathcal{A}} = (\mathcal{A}, \mathbb{E}^A)$ and $\overline{\mathcal{B}} = (\mathcal{B}, \mathbb{E}^B)$ be two L-algebra. Then $\overline{\mathcal{B}}$ is a **subalgebra** of $\overline{\mathcal{A}}$ if algebra $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and $\mathbb{E}^B$ is a restriction of $\mathbb{E}^A$ on B.

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Condition (iii) is equivalent with $\mathbb{E}^A(a, b) \leqslant \theta(a, b)$ for all $a, b \in A$.

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$$\overline{\mathcal{A}}/\theta = (\mathcal{A}/\theta, \mathbb{E}^{\mathcal{A}/\theta}) \quad \text{where} \quad \mathbb{E}^{\mathcal{A}/\theta}([x]_\theta, [y]_\theta) = \theta(x,y)$$

Let $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ be L-algebras of type $\mathcal{T}$. Then a homomorphism $h : A \to B$ from skeleton $\mathcal{A}$ to skeleton $\mathcal{B}$ is a **homomorphism** of $\overline{\mathcal{A}}$ to $\overline{\mathcal{B}}$ if for all $x, y \in A$

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Isomorphism theorems can also be formulated in this framework.

A **direct product** of a family $\{\overline{\mathcal{A}_i} \mid i \in I\}$ of L-algebras of the type $\mathcal{T}$ is an L-algebra

$$\overline{\mathcal{A}} = (\prod_{i \in I} \mathcal{A}_i, \mathbb{E}^{\Pi M_i}),$$

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Subdirect product, subdirect irreducibility can be formulated, and also properties of these in the framework of L algebras, analogously to classical representation theorems.

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Analogously, in semantical sense, a *truth-degree* is associated to formula interpreted in L, where equality relational symbol is modeled by fuzzy equality $\mathbb{E}$.

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Theorem (completeness (Bělohlávek, Vychodil, 2006))

Let L be a complete residuated lattice, X a denumerable set of variables and Σ a set of identities over X. Then for every $t \approx t' \in \Sigma$, $|t \approx t'|_\Sigma = [\![t \approx t']\!]_\Sigma$.

Birkhoff's variety theorem for L-algebras

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Theorem (Bělohlávek, Vychodil, 2006)

*Let L be a complete residuated lattice, $\mathcal{K}$ a class of L-algebras of the same type and X a denumerable set of variables.
Then $\mathcal{K}$ is an equational class if and only if $\mathcal{K}$ is closed under operators H, S and P.*

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Clearly, every congruence on a subalgebra of $\mathcal{A}$ is a weak congruence on $\mathcal{A}$, and vice versa, every nonempty weak congruence θ on $\mathcal{A}$ is a congruence on a subalgebra $\mathcal{B}_\theta$ of $\mathcal{A}$, where $B_\theta := \{x \in A \mid x\,\theta\,x\}$.

The weak congruences on $\mathcal{A}$ form an algebraic lattice under inclusion, denoted by $\mathrm{Con_w}(\mathcal{A})$.

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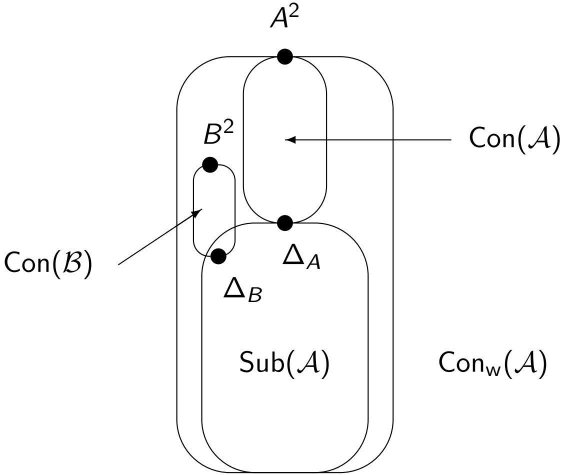
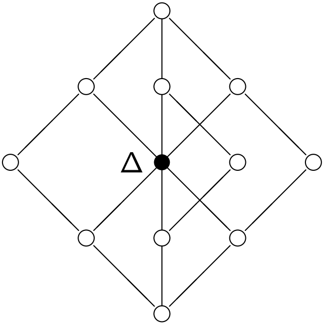
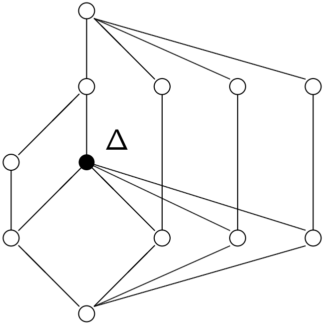
The subalgebra lattice $\mathrm{Sub}(\mathcal{A})$ is isomorphic to the principal ideal generated by Δ, by sending each weak congruence θ contained in Δ to its domain.

Therefore, both the subalgebra lattice and the congruence lattice of an algebra may be recovered and investigated within a single algebraic lattice.

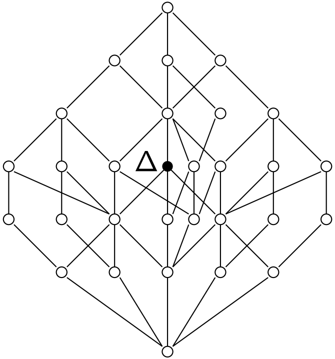

A^2
B^2
$\mathrm{Con}(\mathcal{A})$
$\mathrm{Con}(\mathcal{B})$
Δ_A
Δ_B
$\mathrm{Sub}(\mathcal{A})$
$\mathrm{Con_w}(\mathcal{A})$



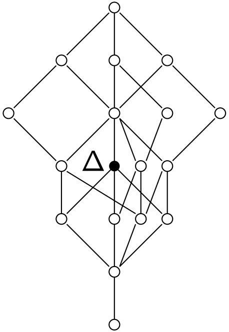
$\mathrm{Con_w}(\mathcal{K})$



$\mathrm{Con_w}(\mathcal{S}_3)$



a) *dihedral group of order 8*



b) *quaternion group*

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Hence, $\mathcal{A}$ *has the CIP if and only if* Δ *is a distributive element of the lattice* $Cw\mathcal{A}$, *if and only if* $n_{\Delta} : \rho \mapsto \rho \vee \Delta$ *is a homomorphism from* $\mathrm{Con}_{\mathrm{w}}(\mathcal{A})$ *onto* $\uparrow\Delta$.

Lattice identities in $\mathrm{Con}_w(\mathcal{A})$

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Proposition

If an algebra $\mathcal{A}$ has the CIP and the CEP, and $\mathrm{Sub}\,\mathcal{A}$ and $\mathrm{Con}\,\mathcal{A}$ are modular (distributive) lattices, then also its lattice of weak congruences is modular (distributive).

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For the converse, observe that in a modular lattice every codistributive element is neutral.

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Theorem

An algebra $\mathcal{A}$ has modular (distributive) lattice of weak congruences if and only if $\mathrm{Sub}\,\mathcal{A}$ and $\mathrm{Con}\,\mathcal{A}$ are modular (distributive) lattices and $\mathcal{A}$ has the CIP and the CEP.

Theorem

The lattice of weak congruences of an algebra $\mathcal{A}$ is relatively complemented if and only if all of the following conditions are satisfied:
- $\mathcal{A}$ has at least one nullary operation,
- no nontrivial congruence on $\mathcal{A}$ has a block which is a subalgebra of $\mathcal{A}$,
- $\mathcal{A}$ satisfies the CEP and the CIP, and
- both $\operatorname{Sub}\mathcal{A}$ and $\operatorname{Con}\mathcal{A}$ are relatively complemented lattices.

Theorem

Let $\mathcal{A}$ be an algebra which has the CIP. Then the weak congruence lattice of $\mathcal{A}$ is complemented if and only if the following conditions hold:
- $\mathcal{A}$ has at least one nullary operation;
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Corollary

The weak congruence lattice of an algebra $\mathcal{A}$ is Boolean if and only if $\mathcal{A}$ satisfies conditions:

(i) for every subalgebra $\mathcal{B}$, $\operatorname{Con}\mathcal{B}$ is isomorphic with $\operatorname{Con}\mathcal{A}$, under $\rho \mapsto \rho_A$ and

(ii) $\operatorname{Sub}\mathcal{A}$ and $\operatorname{Con}\mathcal{A}$ are Boolean lattices.

Theorem (Czédli, Erné, Šešelja, Tepavčević, 2009)

The following statements on a group G are equivalent:

(1) *G is a Dedekind group.*

(2) $\mathrm{Con_w}(G)$ *is modular.*

(3) Δ *is a standard (equivalently, a neutral) element of* $\mathrm{Con_w}(G)$.

(4) *G has the* CIP *and the* CEP.

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Easily solved by Grätzer-Schmidt theorem:
Let $\mathcal{B} = (A, F)$ be an algebra such that $\mathrm{Con}\,\mathcal{B}$ is isomorphic with L. Then the required algebra $\mathcal{A}$ can be obtained by adding to F all the elements from A as nullary operations: $\mathcal{A} = (A, F \cup \{A\})$.
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The above construction by which the diagonal relation of the algebra corresponds to the bottom of the lattice is called the **trivial representation**.

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A representation by which the diagonal relation corresponds to an element different from the bottom of the lattice is said to be **non-trivial**.

Ω-sets and structures

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Observe that *separated symmetric and transitive relation on a set A is the equality relation on a subset of A.*

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Proposition

The collection $\{\mu_p \mid p \in \Omega\}$ of all cuts of the function $\mu : X \to \Omega$ is a closure system on X.

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For every n-ary operation $f \in F$, for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, and for every constant (nullary operation) $c \in F$

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In addition, the collection of all cuts $\{E_p \mid p \in \Omega\}$ of E is a closure system, a subposet of the lattice of all weak equivalences on A.

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Theorem (Šešelja, Tepavčević, unpublished)

Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\mathrm{Con_w}(\mathcal{A})$ such that for every $a \in A$,

$$\bigcap\{R \in \mathcal{R} \mid (a, a) \in R\} \subseteq \Delta_A.$$

Then there is a complete lattice Ω and an Ω-algebra $(\mathcal{A}, E)$ with the underlying algebra $\mathcal{A}$, such that $\mathcal{R}$ consists of cuts of E.

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Then, $(\mathcal{A}, E)$ **satisfies identity** $u \approx v$ (this identity **holds** on $(\mathcal{A}, E)$) if the following condition is fulfilled:

Identities

Let

$$u(x_1, \ldots, x_n) \approx v(x_1, \ldots, x_n) \quad (\text{briefly} \quad u \approx v)$$

be an identity in the type of an Ω-algebra $(\mathcal{A}, E)$. We assume that variables appearing in terms u and v are from $x_1, \ldots, x_n$.

Then, $(\mathcal{A}, E)$ **satisfies identity** $u \approx v$ (this identity **holds** on $(\mathcal{A}, E)$) if the following condition is fulfilled:

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant E(u(a_1, \ldots, a_n), v(a_1, \ldots, a_n)),$$

for all $a_1, \ldots, a_n \in A$ and the term-operations corresponding to terms u and v respectively.

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Proposition

If an identity $u \approx v$ holds on an algebra $\mathcal{A}$, then it also holds on an Ω-algebra $(\mathcal{A}, E)$.

Theorem (B. and V. Budimirović, Šešelja,Tepavčević, 2016)

Let $(\mathcal{A}, E)$ be an Ω-algebra, and $\mathcal{F}$ a set of identities in the language of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ satisfies (all identities in) $\mathcal{F}$ if and only if for every $p \in L$ the quotient algebra μ_p/E_p satisfies the same identities.

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In addition, the poset

$$(\{\mu_p/E_p \mid p \in \Omega\}, \subseteq)$$

is a closure system which is, up to an isomorphism, a subposet of the weak congruence lattice of $\mathcal{A}$.

Corollary (Šešelja, Tepavčević, unpublished)

Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\mathrm{Con_w}(\mathcal{A})$ such that for every $a \in A$, $\bigcap\{R \in \mathcal{R} \mid (a, a) \in R\} \subseteq \Delta_A$. Let also $\mathcal{F}$ be a set of identities in the language of $\mathcal{A}$ and suppose that for every $\rho \in \mathcal{R}$, the algebra $\mathrm{dom}\rho/\rho$ fulfills these identities.
Then there is a complete lattice Ω and an Ω-algebra $(\mathcal{A}, E)$ with the underlying algebra $\mathcal{A}$, such that $\mathcal{R}$ consists of cuts of E and $(\mathcal{A}, E)$ satisfies $\mathcal{F}$.

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$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z,$$

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In terms of Ω-algebras, these identities are equivalent with formulas:

(i) $E(x \cdot (y \cdot z), (x \cdot y) \cdot z) \geqslant \mu(x) \wedge \mu(y) \wedge \mu(z)$,
(ii) $E(x \cdot e, x) \geqslant \mu(x)$ and $E(e \cdot x, x) \geqslant \mu(x)$,
(iii) $E(x \cdot x^{-1}, e) \geqslant \mu(x)$ and $E(x^{-1} \cdot x, e) \geqslant \mu(x)$.

Theorem

Let $(\mathcal{G}, E)$ be an Ω-algebra. Then, $(\mathcal{G}, E)$ is an Ω-group if and only if for every $p \in \Omega$, the quotient structure μ_p/E_p is a group.

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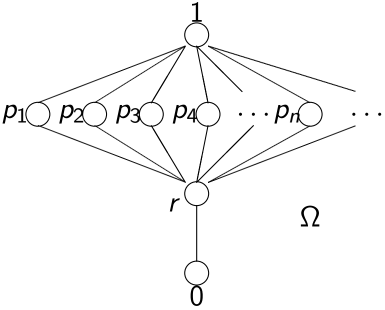

$$\mu := \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \dots & \mathbf{n} & \dots \\ 1 & p_1 & p_2 & p_3 & \dots & p_n & \dots \end{pmatrix}.$$

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E^μ	**0**	**1**	**2**	**3**	**4**	**5**	...
0	1	0	r	0	r	0	...
1	0	p_1	0	r	0	r	...
2	r	0	p_2	0	r	0	...
3	0	r	0	p_3	0	r	...
4	r	0	r	0	p_4	0	...
5	0	r	0	r	0	p_5	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	

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2	r	0	p_2	0	r	0	···
3	0	r	0	p_3	0	r	···
4	r	0	r	0	p_4	0	···
5	0	r	0	r	0	p_5	···
⋮	⋮	⋮	⋮	⋮	⋮	⋮	

The structure $(\mathcal{G}, E^\mu)$ is an Ω-group.

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;

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$\mathbf{0}$	1	0
$\mathbf{n}$	0	1

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$\oplus$	**0**	**n**
0	**0**	**n**
n	**n**	**0**

;

$E^{\mu}_{p_n}$	**0**	**n**
0	1	0
n	0	1

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For every $p_n \in \Omega$, the quotient structure $\mu_{p_n}/E^{\mu}_{p_n}$ is a two-element group.

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f	f	e	h	g	k	j
g	g	j	e	k	f	h
h	h	k	f	j	e	g
j	j	g	k	e	h	f
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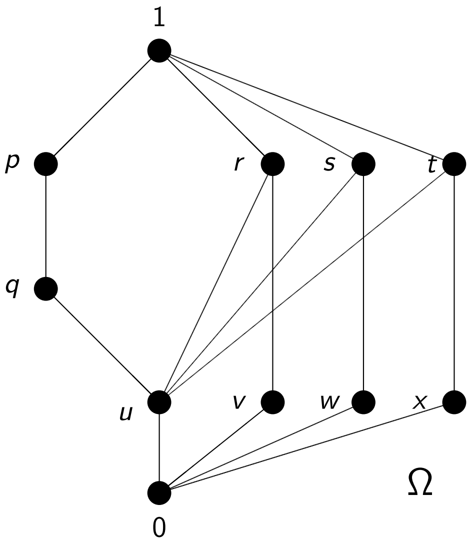
The lattice Ω (which is not a Heyting algebra) is presented by the diagram.


1
p
r
s
t
q
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v
w
x
0
Ω

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e	1	x	w	q	q	v
f	x	t	u	0	0	u
g	w	u	s	0	0	u
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All the structures μ_z / E_z^μ, $z \in \Omega$ are groups of order 3, 2 or 1, hence Abelian.

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Therefore, this structure is an Abelian Ω-group, identity $x \cdot y \approx y \cdot x$ holds as the formula

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$$\mu(x) \wedge \mu(y) \leqslant E^\mu(x \cdot y, y \cdot x).$$

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k	0	0	0	0	0	0 .

Hence, E_p^μ is a weak congruence on S_3, a diagonal of $\mu_p = \{e, h, j\}$ and μ_p/E_p^μ is a group of order 3.

$$\mu_u = \begin{pmatrix} e & f & g & h & j & k \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \text{hence } \mu_u \text{ is the underlying group } S_3.$$

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$\mu_u/E_u^\mu = \{\{e, h, j\}, \{f, g, h\}\}$ i.e., it is is a two-element group, similarly for other cuts.

Relational structures: Ω-poset and Ω-lattice

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Proposition

Let (M, E, R) be an Ω-poset. Then for every $p \in \Omega$, the binary relation $\leq_p$ on μ_p/E_p, defined by
$[x]_{E_p} \leq_p [y]_{E_p}$ *if and only if* $(x,y) \in R_p$
is a classic ordering relation.

Let (M, E, R) be an Ω-poset and $a, b \in M$. An element $c \in M$ is a **pseudo-infimum** of a and b, if for every $p \leqslant \mu(a) \wedge \mu(b)$ the following holds:

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(i) $p \leqslant R(c, a) \wedge R(c, b)$ and
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An element $d \in M$ is a **pseudo-supremum** of $a, b \in M$, if for every $p \leqslant \mu(a) \wedge \mu(b)$ the following holds:

Let (M, E, R) be an Ω-poset and $a, b \in M$. An element $c \in M$ is a **pseudo-infimum** of a and b, if for every $p \leqslant \mu(a) \wedge \mu(b)$ the following holds:
(i) $p \leqslant R(c, a) \wedge R(c, b)$ and
for every $x \in \mu_p$
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It is straightforward that a pseudo-infimum (supremum) of a and b belongs to μ_p for every $p \leqslant \mu(a) \wedge \mu(b)$.

A pseudo-infimum and a pseudo-supremum for given $a, b \in M$, if they exist, are not unique in general.

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Proposition

Let (M, E, R) be an Ω-poset and $a, b, c, c_1, d, d_1 \in M$. If c is a pseudo-infimum of a and b, then $\mu(a) \wedge \mu(b) \leqslant E(c, c_1)$ if and only if c_1 is also a pseudo-infimum of a and b. Analogously, if d is a pseudo-supremum of a and b, then $\mu(a) \wedge \mu(b) \leqslant E(d, d_1)$ if and only if d_1 is also a pseudo-supremum of a and b.

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Since for $p \leqslant q$, every equivalence class of μ_q/E_q is contained in a class of μ_p/E_p, we get that pseudo-infima (suprema) of two elements a, b, if they exist, belong to the same equivalence class in μ_p/E_p, for $p \leqslant \mu(a) \wedge \mu(b)$.

We say that an Ω-poset (M, E, R) is an **Ω-lattice as an ordered structure**, if for every $a, b \in M$ there exist a pseudo-infimum and a pseudo-supremum.

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Theorem (Edeghagba, Šešelja, Tepavčević, 2017)

Let (M, E, R) be an Ω-poset. Then it is an Ω-lattice as an ordered structure if and only if for every $q \in \Omega$, the poset $(\mu_q/E_q, \leq_q)$ is a lattice, and the following holds:
for all $a, b \in M$, and $p = \mu(a) \wedge \mu(b)$,
$\inf([a]_{E_p}, [b]_{E_p}) \subseteq \inf([a]_{E_q}, [b]_{E_q})$ *and*
$\sup([a]_{E_p}, [b]_{E_p}) \subseteq \sup([a]_{E_q}, [b]_{E_q})$,
for every q, $q \leqslant p$.

Ω-lattice as Ω-algebra

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Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid and $E : M^2 \to \Omega$ an Ω-valued equality on M, hence (M, E) is supposed to be an Ω-set.

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$E(x, y) \wedge E(z, t) \leqslant E(x \sqcap z, y \sqcap t)$ and
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Proposition

If E is a compatible Ω-valued equality on a bi-groupoid $\mathcal{M} = (M, \sqcap, \sqcup)$, and $\mu : M \to \Omega$ is defined by $\mu(x) = E(x, x)$, then the following hold:
(i) For all $x, y \in M$,
$\mu(x) \wedge \mu(y) \leqslant \mu(x \sqcap y)$ *and* $\mu(x) \wedge \mu(y) \leqslant \mu(x \sqcup y)$.
(ii) For every $p \in \Omega$, the cut μ_p of μ is a sub-bi-groupoid of $\mathcal{M}$.
(iii) For every $p \in \Omega$, the cut E_p of E is a congruence on μ_p.

Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid and $(\mathcal{M}, E)$ an Ω-algebra. Then $(\mathcal{M}, E)$ is an **Ω-lattice as an Ω-algebra** (Ω-lattice as an algebra), if it satisfies the lattice identities:

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$$
\begin{aligned}
&\ell 1: \ x \sqcap y \approx y \sqcap x \\
&\ell 2: \ x \sqcup y \approx y \sqcup x
\end{aligned}
\qquad (commutativity)
$$

$$
\begin{aligned}
&\ell 3: \ x \sqcap (y \sqcap z) \approx (x \sqcap y) \sqcap z \\
&\ell 4: \ x \sqcup (y \sqcup z) \approx (x \sqcup y) \sqcup z
\end{aligned}
\qquad (associativity)
$$

$$
\begin{aligned}
&\ell 5: \ (x \sqcap y) \sqcup x \approx x \\
&\ell 6: \ (x \sqcup y) \sqcap x \approx x.
\end{aligned}
\qquad (absorption)
$$

In terms of Ω-algebras for all $x, y, z \in M$, the following formulas should be satisfied, where, as already indicated, the mapping $\mu : M \to \Omega$ is defined by $\mu(x) = E(x, x)$:

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\begin{aligned}
L1 &: \ \mu(x) \wedge \mu(y) \leqslant E(x \sqcap y, y \sqcap x) \\
L2 &: \ \mu(x) \wedge \mu(y) \leqslant E(x \sqcup y, y \sqcup x) \\
L3 &: \ \mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \sqcap y) \sqcap z, x \sqcap (y \sqcap z)) \\
L4 &: \ \mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \sqcup y) \sqcup z, x \sqcup (y \sqcup z)) \\
L5 &: \ \mu(x) \wedge \mu(y) \leqslant E((x \sqcap y) \sqcup x, x) \\
L6 &: \ \mu(x) \wedge \mu(y) \leqslant E((x \sqcup y) \sqcap x, x).
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L3 &: \ \mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \sqcap y) \sqcap z, x \sqcap (y \sqcap z)) \\
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L6 &: \ \mu(x) \wedge \mu(y) \leqslant E((x \sqcup y) \sqcap x, x).
\end{aligned}$$

Theorem

[Edeghagba, Šešelja, Tepavčević, 2017] Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid, and let E be an Ω-valued compatible equality on $\mathcal{M}$. Then, $(\mathcal{M}, E)$ is an Ω-lattice if and only if for every $p \in \Omega$, the quotient structure μ_p / E_p is a lattice.

Equivalence of two approaches

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For every pair a, b of elements from M, $a \sqcap b$ is an arbitrary, fixed pseudo-infimum of a and b, and $a \sqcup b$ is an arbitrary, fixed pseudo-supremum of a and b.

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Theorem (Edeghagba, Šešelja, Tepavčević, 2017)

If (M, E, R) is an Ω-lattice as an ordered structure, and $\mathcal{M} = (M, \sqcap, \sqcup)$ the bi-groupoid in which operations $\sqcap$, $\sqcup$ are introduced above, then $(\mathcal{M}, E)$ is an Ω-lattice as an algebra.

Theorem (Edeghagba, Šešelja, Tepavčević, 2017)

Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid, $(\mathcal{M}, E)$ an Ω-lattice as an algebra and $R : M^2 \to \Omega$ an Ω-valued relation on M defined by $R(x, y) := \mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x)$.
Then, (M, E, R) is an Ω-lattice as an ordered structure.

Example

Example

Let $M = \{a, b, c, d, e, f, g\}$, and let Ω be the lattice given in Figure 1.

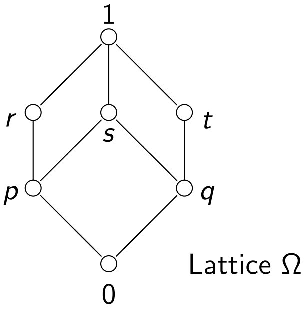

Figure 1

E	a	b	c	d	e	f	g
a	r	p	0	0	0	0	0
b	p	r	0	0	0	0	0
c	0	0	s	q	q	0	0
d	0	0	q	1	q	0	0
e	0	0	q	q	1	0	0
f	0	0	0	0	0	q	0
g	0	0	0	0	0	0	q

Table 1: Ω-valued equality E

R	a	b	c	d	e	f	g
a	r	r	0	0	r	0	0
b	p	r	0	0	r	0	0
c	0	0	s	q	s	q	q
d	r	r	s	1	1	q	q
e	0	0	q	q	1	q	q
f	0	0	0	0	0	q	q
g	0	0	0	0	0	0	q

Table 2: Ω-valued order R

E	a	b	c	d	e	f	g
a	r	p	0	0	0	0	0
b	p	r	0	0	0	0	0
c	0	0	s	q	q	0	0
d	0	0	q	1	q	0	0
e	0	0	q	q	1	0	0
f	0	0	0	0	0	q	0
g	0	0	0	0	0	0	q

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e	0	0	q	q	1	q	q
f	0	0	0	0	0	q	q
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Table 2: Ω-valued order R

$$E(x,y) = R(x,y) \wedge R(y,x)$$

E	a	b	c	d	e	f	g
a	r	p	0	0	0	0	0
b	p	r	0	0	0	0	0
c	0	0	s	q	q	0	0
d	0	0	q	1	q	0	0
e	0	0	q	q	1	0	0
f	0	0	0	0	0	q	0
g	0	0	0	0	0	0	q

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R	a	b	c	d	e	f	g
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e	0	0	q	q	1	q	q
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g	0	0	0	0	0	0	q

Table 2: Ω-valued order R

$$E(x,y) = R(x,y) \wedge R(y,x)$$

(M, E, R) is an Ω-lattice as an ordered structure.

$$\mu = \begin{pmatrix} a & b & c & d & e & f & g \\ r & r & s & 1 & 1 & q & q \end{pmatrix}.$$

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The cuts of μ and the cuts of E represented by partitions are:

$$\begin{array}{ll}
\mu_0 = M\ ; & E_0 = M^2; \\
\mu_p = \{a, b, c, d, e\}\ ; & E_p = \{\{a, b\}, \{c\}, \{d\}, \{e\}\}; \\
\mu_q = \{c, d, e, f, g\}\ ; & E_q = \{\{c, d, e\}, \{f\}, \{g\}\}; \\
\mu_r = \{a, b, d, e\}\ ; & E_r = \{\{a\}, \{b\}, \{d\}, \{e\}\}; \\
\mu_s = \{c, d, e\}\ ; & E_s = \{\{c\}, \{d\}, \{e\}\}; \\
\mu_t = \mu_1 = \{d, e\}; & E_t = E_1 = \{\{d\}, \{e\}\}.
\end{array}$$

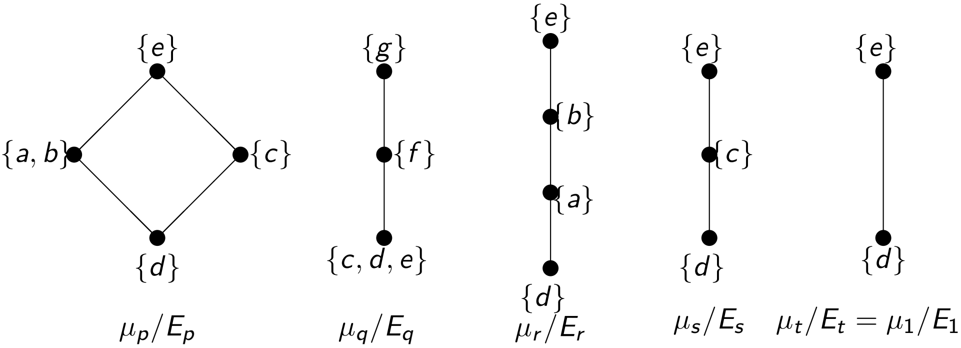

Figure 2: Quotient lattices

Two binary operations on M are constructed by means of pseudo-infima and pseudo-suprema. In this way, we obtain the bi-groupoid $\mathcal{M} = (M, \sqcap, \sqcup)$.

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$\sqcap$	a	b	c	d	e	f	g
a	a	a	d	d	a	b^{**}	c^{**}
b	a	b	d	d	b	a^{**}	g^{**}
c	d	d	c	d	c	c^{*}	c^{*}
d	d	d	d	d	d	d^{*}	d^{*}
e	a	b	c	d	e	e^{*}	c^{*}
f	d^{**}	a^{**}	d^{*}	e^{*}	c^{*}	f	f
g	a^{**}	e^{**}	c^{*}	e^{*}	c^{*}	f	g

$\sqcup$	a	b	c	d	e	f	g
a	a	b	e	a	e	f^{**}	a^{**}
b	b	b	e	b	e	a^{**}	c^{**}
c	e	e	c	c	e	f	g
d	a	b	c	d	e	f	g
e	e	e	e	e	e	f	g
f	g^{**}	g^{**}	f	f	f	f	g
g	b^{**}	g^{**}	g	g	g	g	g

$\sqcup$	a	b	c	d	e	f	g
a	a	b	e	a	e	f^{**}	a^{**}
b	b	b	e	b	e	a^{**}	c^{**}
c	e	e	c	c	e	f	g
d	a	b	c	d	e	f	g
e	e	e	e	e	e	f	g
f	g^{**}	g^{**}	f	f	f	f	g
g	b^{**}	g^{**}	g	g	g	g	g

$(\mathcal{M}, E)$ is an Ω-lattice as an algebra.

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A groupoid $(Q, \cdot)$ is a **quasigroup** if for all $a, b \in Q$, linear equations: $a \cdot x = b$ and $y \cdot a = b$ are uniquely solvable for x, y.

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An **equasigroup** is an algebra $(Q, \cdot, \backslash, /)$ which satisfies the following identities:

$$Q1: \; y = x \cdot (x \backslash y);$$
$$Q2: \; y = x \backslash (x \cdot y);$$
$$Q3: \; y = (y/x) \cdot x;$$
$$Q4: \; y = (y \cdot x)/x.$$

Theorem

If $(Q, \cdot)$ *is a quasigroup, then* $(Q, \cdot, \backslash, /)$ *is an equasigroup, where the additional binary operations* $\backslash$ *and* $/$ *are defined by:*

$$a \backslash b = c \quad \text{iff} \quad b = a \cdot c \qquad \text{and} \qquad a/b = c \quad \text{iff} \quad a = c \cdot b.$$

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Alternatively, an equasigroup is an **eloop** if for all x, y, $x\backslash x = y/y$; in this approach $x\backslash x$ serves as the identity element.

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Finally, a **group** is an associative loop.
Here we consider groups in the language with a binary operation $\cdot$, unary operation $^{-1}$ and a constant e.

Ω-groupoid, Ω-quasigroup

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Analogously, an equation $y \cdot a = b$ is **solvable over** $(\mathcal{Q}, E)$ if there is $d \in Q$ such that

$$\mu(a) \wedge \mu(b) \leqslant \mu(d) \wedge E(d \cdot a, b).$$

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If c is a solution of the equation $a \cdot x = b$ over $(\mathcal{Q}, E)$ and $c_1 \in Q$ fulfills $E(a \cdot c_1, b) \geqslant p$ for some $p \leqslant \mu(a) \wedge \mu(b)$, then

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Analogously, if d is a solution of the equation $y \cdot a = b$ over $(\mathcal{Q}, E)$ and $d_1 \in Q$ fulfills $E(d_1 \cdot a, b) \geqslant p$ for some $p \leqslant \mu(a) \wedge \mu(b)$, then

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Hence, an E-uniquely solvable equation may have several solutions. All these solutions are equal up to the Ω-equality E. More precisely, we have the following.

Theorem (Krapež, Šešelja, Tepavčević, (submitted))

Let $(\mathcal{Q}, E)$ be an Ω-groupoid. If equations $a \cdot x = b$ and $y \cdot a = b$, are E-uniquely solvable over $(\mathcal{Q}, E)$ for all $a, b \in Q$, then for every $p \in L$ the quotient groupoid μ_p / E_p is a quasigroup.

We say that an Ω-groupoid $(\mathcal{Q}, E)$ is an **Ω-quasigroup**, if every equation of the form $a \cdot x = b$ or $y \cdot a = b$ is E-uniquely solvable over $(\mathcal{Q}, E)$.

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Let $(\mathcal{Q}, E)$ be an Ω-groupoid. If for all $a, b \in Q$ and for every $p \leqslant \mu(a) \wedge \mu(b)$ the quotient groupoid μ_p/E_p is a quasigroup, then $(\mathcal{Q}, E)$ is an Ω-quasigroup.

Ω-equasigroup

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Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be an algebra in the language with three binary operations, L a complete lattice and $E : Q^2 \rightarrow L$ an Ω-valued compatible equality over $\mathcal{Q}$.

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Then, $(\mathcal{Q}, E)$ is an **Ω-equasigroup**, if identities $Q1, \dots, Q4$ hold:

$Q1: \; y = x \cdot (x \backslash y);$

$Q2: \; y = x \backslash (x \cdot y);$

$Q3: \; y = (y / x) \cdot x;$

$Q4: \; y = (y \cdot x) / x.$

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This means that the following formulas should be satisfied, where $\mu : Q \to L$ is defined by $\mu(x) = E(x, x)$:
$QE1: \; \mu(x) \wedge \mu(y) \leqslant E(y, x \cdot (x \backslash y));$
$QE2: \; \mu(x) \wedge \mu(y) \leqslant E(y, x \backslash (x \cdot y));$
$QE3: \; \mu(x) \wedge \mu(y) \leqslant E(y, (y / x) \cdot x);$
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If $((Q, \cdot, \backslash, /), E)$ is an Ω-equasigroup, then for every $p \in L$, the quotient structure μ_p / E_p is a classical equasigroup.

Theorem (Krapež, Šešelja, Tepavčević, (submitted))

If $((Q,\cdot,\backslash,/),E)$ *is an* Ω*-equasigroup, then for every* $p\in L$*, the quotient structure* μ_p/E_p *is a classical equasigroup.*

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If $((Q,\cdot,\backslash,/),E)$ *is an* Ω*-equasigroup, then* $((Q,\cdot),E)$ *is an* Ω*-quasigroup.*

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If $((Q,\cdot,\backslash,/),E)$ is an Ω-equasigroup, then $((Q,\cdot),E)$ is an Ω-quasigroup.

The converse follows by the Axiom of Choice (AC).

Let $((Q, \cdot), E)$ be an Ω-groupoid which is an Ω-quasigroup.

Let $((Q, \cdot), E)$ be an Ω-groupoid which is an Ω-quasigroup. Then, for every $p \in L$, the quotient groupoid $(\mu_p/E_p, \cdot)$ is a quasigroup, where the operation $\cdot$ is defined by $[a]_{E_p} \cdot [b]_{E_p} = [a \cdot b]_{E_p}$, $a, b \in \mu_p$.

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Therefore, the structure $(\mu_p/E_p, \cdot, \backslash, /)$ is an equasigroup, where the operations $\backslash$ and $/$ are the usual ones:

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Therefore, the structure $(\mu_p/E_p, \cdot, \backslash, /)$ is an equasigroup, where the operations $\backslash$ and $/$ are the usual ones:

$$[a]_{E_p} \backslash [b]_{E_p} = [c]_{E_p} \text{ if and only if } [a]_{E_p} \cdot [c]_{E_p} = [b]_{E_p},$$
$$[b]_{E_p} / [a]_{E_p} = [d]_{E_p} \text{ if and only if } [d]_{E_p} \cdot [a]_{E_p} = [b]_{E_p}.$$

Let $((Q,\cdot),E)$ be an Ω-groupoid which is an Ω-quasigroup. Then, for every $p\in L$, the quotient groupoid $(\mu_p/E_p,\cdot)$ is a quasigroup, where the operation $\cdot$ is defined by
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Therefore, the structure $(\mu_p/E_p,\cdot,\backslash,/)$ is an equasigroup, where the operations $\backslash$ and $/$ are the usual ones:

$$[a]_{E_p}\backslash[b]_{E_p}=[c]_{E_p} \text{ if and only if } [a]_{E_p}\cdot[c]_{E_p}=[b]_{E_p},$$
$$[b]_{E_p}/[a]_{E_p}=[d]_{E_p} \text{ if and only if } [d]_{E_p}\cdot[a]_{E_p}=[b]_{E_p}.$$

Let us define binary operations $\backslash$ and $/$ over Q in the following way:

Let $((Q,\cdot),E)$ be an Ω-groupoid which is an Ω-quasigroup. Then, for every $p \in L$, the quotient groupoid $(\mu_p/E_p,\cdot)$ is a quasigroup, where the operation $\cdot$ is defined by
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Let us define binary operations $\backslash$ and $/$ over Q in the following way:

For every pair $a,b\in Q$, $a\backslash b=c$, where c is an element chosen by AC from $[a]_{E_p}\backslash[b]_{E_p}$ in the quasigroup μ_p/E_p, where $p=\mu(a)\wedge\mu(b)$. Analogously, $b/\,a=d$, where d is chosen by the AC from $[b]_{E_p}/\,[a]_{E_p}$ in μ_p/E_p, for $p=\mu(a)\wedge\mu(b)$.

Lemma

Let $((Q, \cdot), E)$ *be an* Ω*-groupoid which is an* Ω*-quasigroup. Then the operations* $\backslash$ *and* $/$ *over* Q *are well defined.*

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Theorem (Krapež, Šešelja, Tepavčević, (submitted))

Let $((Q,\cdot),E)$ *be an* Ω*-groupoid which is an* Ω*-quasigroup. Then the structure* $((Q,\cdot,\backslash,/),E)$ *is an* Ω*-equasigroup, where the binary operations* $\backslash$ *and* $/$ *over* Q *are defined by Axiom of Choice as above.*

Example

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$\cdot$	a	b	c	d	e
a	b	c	a	a	e
b	a	b	c	d	e
c	c	a	b	b	e
d	d	a	b	b	e
e	e	e	e	e	a

Table 1

Example

$\cdot$	a	b	c	d	e
a	b	c	a	a	e
b	a	b	c	d	e
c	c	a	b	b	e
d	d	a	b	b	e
e	e	e	e	e	a

Table 1

Let $(Q, \cdot)$ be a groupoid given in Table 1.

Example

$\cdot$	a	b	c	d	e
a	b	c	a	a	e
b	a	b	c	d	e
c	c	a	b	b	e
d	d	a	b	b	e
e	e	e	e	e	a

Table 1

Let $(Q, \cdot)$ be a groupoid given in Table 1.

This is not a quasigroup, e.g., equation $a \cdot x = d$ does not have a solution in Q.

The lattice L is given by the diagram in Figure 1:

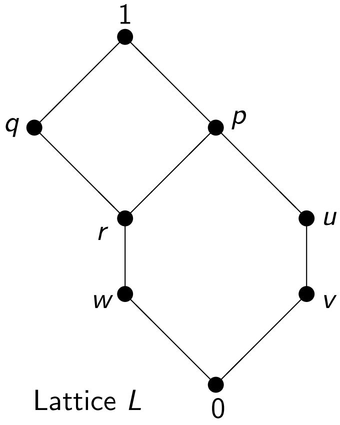

Figure 1

An Ω-valued equality is presented by Table 2.

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E	a	b	c	d	e
a	1	p	p	r	v
b	p	1	p	r	v
c	p	p	1	q	v
d	r	r	q	q	0
e	v	v	v	0	u

Table 2

An Ω-valued equality is presented by Table 2.

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a	1	p	p	r	v
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Table 2

The function $\mu : Q \to L$ ($\mu(x) = E(x, x)$ for all $x \in Q$):

$$\mu = \begin{pmatrix} a & b & c & d & e \\ 1 & 1 & 1 & q & u \end{pmatrix}.$$

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$\mu_p/E_p = \{\{a, b, c\}\}$,
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E.g., the mentioned equation $a \cdot x = d$ which does not have a classical solution in Q, possesses a solution with respect to fuzzy equality E.
Indeed, due to $\mu(a) \wedge \mu(d) = q$, this solution is element b, since the class $X = \{b\}$ is the unique solution of the equation $[a]_{E_q} \cdot X = [d]_{E_q}$ over the quasigroup μ_q/E_q (observe that $[d]_{E_q} = \{c, d\}$).

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$$\mu(a) \wedge \mu(d) = q \leqslant \mu(b) \wedge E(a \cdot b, d) = \mu(b) \wedge E(c, d) = 1 \wedge q = q.$$

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$$\mu(a) \wedge \mu(d) = q \leqslant \mu(b) \wedge E(a \cdot b, d) = \mu(b) \wedge E(c, d) = 1 \wedge q = q.$$

Hence, $a \cdot b$ and d are E-equal with grade q.

Ω-loop and Ω-group

Ω-loop and Ω-group

An **Ω-loop** is an Ω-algebra $(\mathcal{Q}, E)$, where $\mathcal{Q} = (Q, \cdot, e)$ is a structure with a binary operation $\cdot$ and a constant e, $((Q, \cdot), E)$ is an Ω-quasigroup, $E(e, e) = 1$ and the formula $LG2$ holds.

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An **Ω-semigroup** is an Ω-algebra $((Q, \cdot), E)$ where $(Q, \cdot)$ is a groupoid and the formula $LG1$ holds.

The proof of the following theorem depends on the Axiom of Choice (AC).

Theorem (Krapež, Šešelja, Tepavčević, (submitted))

Let $((Q, \cdot, e), E)$ be an Ω-algebra. There is a unary operation $^{-1}$ on Q such that $((Q, \cdot, {}^{-1}, e), E)$ is an Ω-group if and only if $((Q, \cdot), E)$ is an Ω-semigroup and $((Q, \cdot, e), E)$ an Ω-loop.

Theorem (Krapež, Šešelja, Tepavčević, (submitted))

Let $\mathcal{Q} = (Q, \cdot)$ be an arbitrary groupoid, let a, b be particular elements in Q, and let $E : Q^2 \to L$ be an Ω-valued equality over $\mathcal{Q}$. Then the equation $a \cdot x = b$ has a unique solution w.r.t. E, if the equation $[a]_{E_p} \cdot X = [b]_{E_p}$, for $p = \mu(a) \wedge \mu(b)$, has a (classical) unique solution in the quotient groupoid $(\mu_p/E_p, \cdot)$.

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Our approach gives new directions in producing new cryptosystems.

Although our basic structure is not a quasigroup, quotients of cuts are, and similar techniques from this field can be applied.

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Finally, in *control systems* which are usually designed by lattice-valued relations, it could be possible to apply Ω-valued ordered structures in order to get more sensitive coordination among input and output signals.

References

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Thank you for your attention!