Monotonicity and continuity of weighted densities

by

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Main definitions

$$A \subseteq \mathbb{N}^*, \alpha \geq -1$$

Lower and upper α-densities of A

Defined respectively as

$$\underline{d}_\alpha A = \liminf_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)},$$

$$\overline{d}_\alpha A = \limsup_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)},$$

where

$$A_\alpha(n) = \sum_{\substack{a\in A\\ a\leq n}} a^\alpha, \qquad \mathbb{N}_\alpha(n) = \sum_{k=1}^{n} k^\alpha.$$

Remark

These densities are *weighted* densities in that they attach a *weight* equal to k^α to the integer k.

Monotonicity of α-densities

The following result holds

THEOREM

Let α, β be real numbers verifying the inequalities $-1 \leq \alpha < \beta$. We have

$$0 \leq \underline{d}_{\beta} A \leq \underline{d}_{\alpha} A \leq \overline{d}_{\alpha} A \leq \overline{d}_{\alpha} A \leq 1.$$

Remark

This Theorem generalizes the well known inequality

$$0 \leq \underline{d} A \leq \underline{\delta} A \leq \overline{\delta} A \leq \overline{d} A \leq 1$$

between the asymptotic densities $\underline{d}A$ and $\overline{d}A$ ($\beta = 0$) and the logarithmic densities $\underline{\delta}A$ and $\overline{\delta}A$ ($\alpha = -1$).The reference for this particular case is [H-R].

A is fixed throughout.

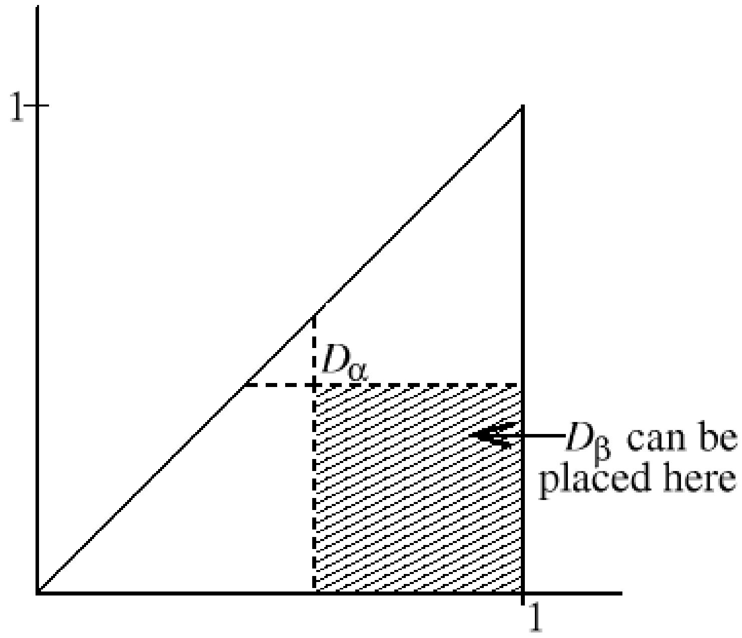
Consider α and β, with $-1 \leq \alpha < \beta$.

Put

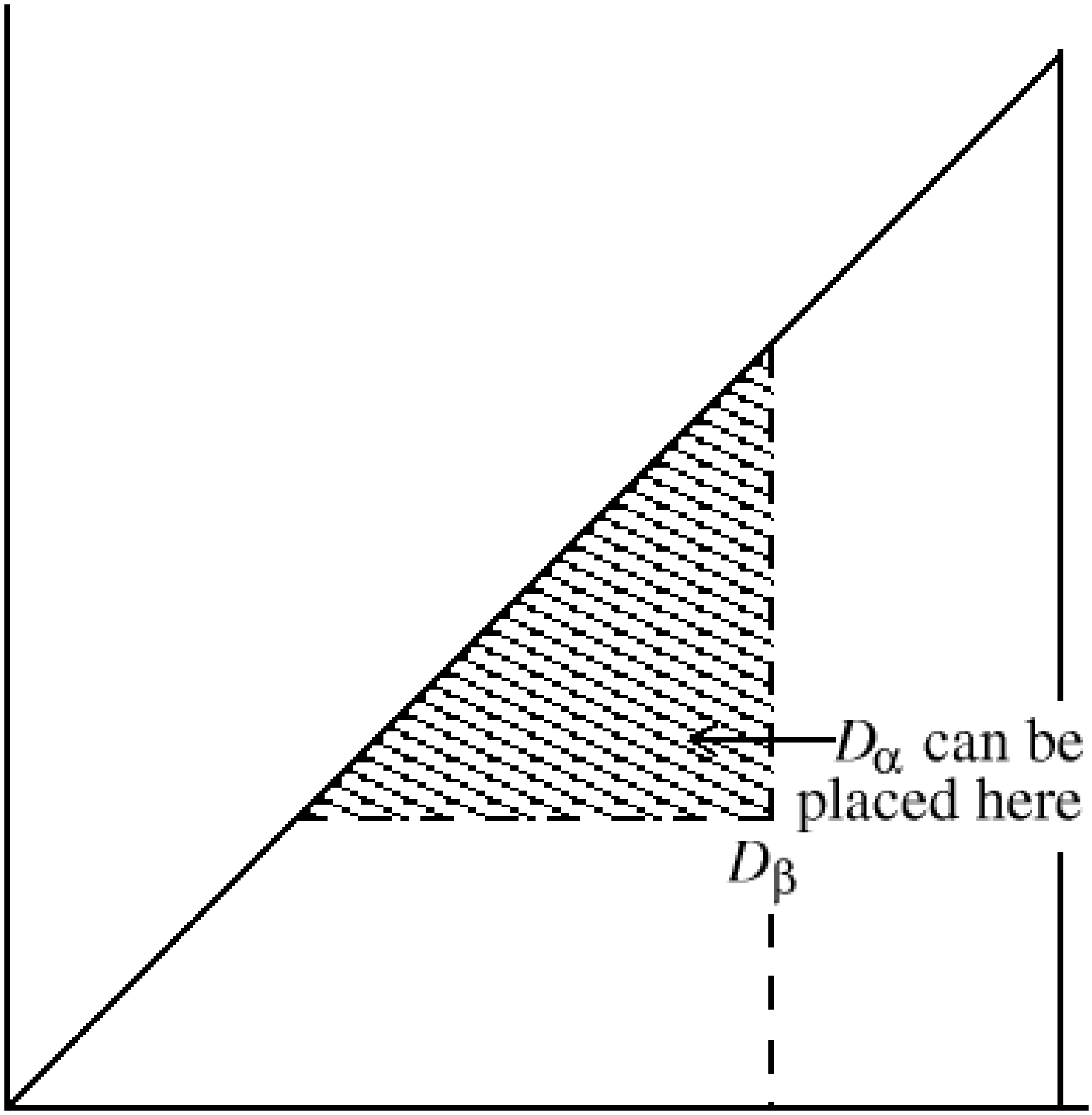
$$D_\alpha = (\underline{d}_\alpha A, \overline{d}_\alpha A) \qquad D_\beta = (\underline{d}_\beta A, \overline{d}_\beta A)$$

We want to make a picture of the reciprocal positions of D_α and D_β in the (x, y)-plane.

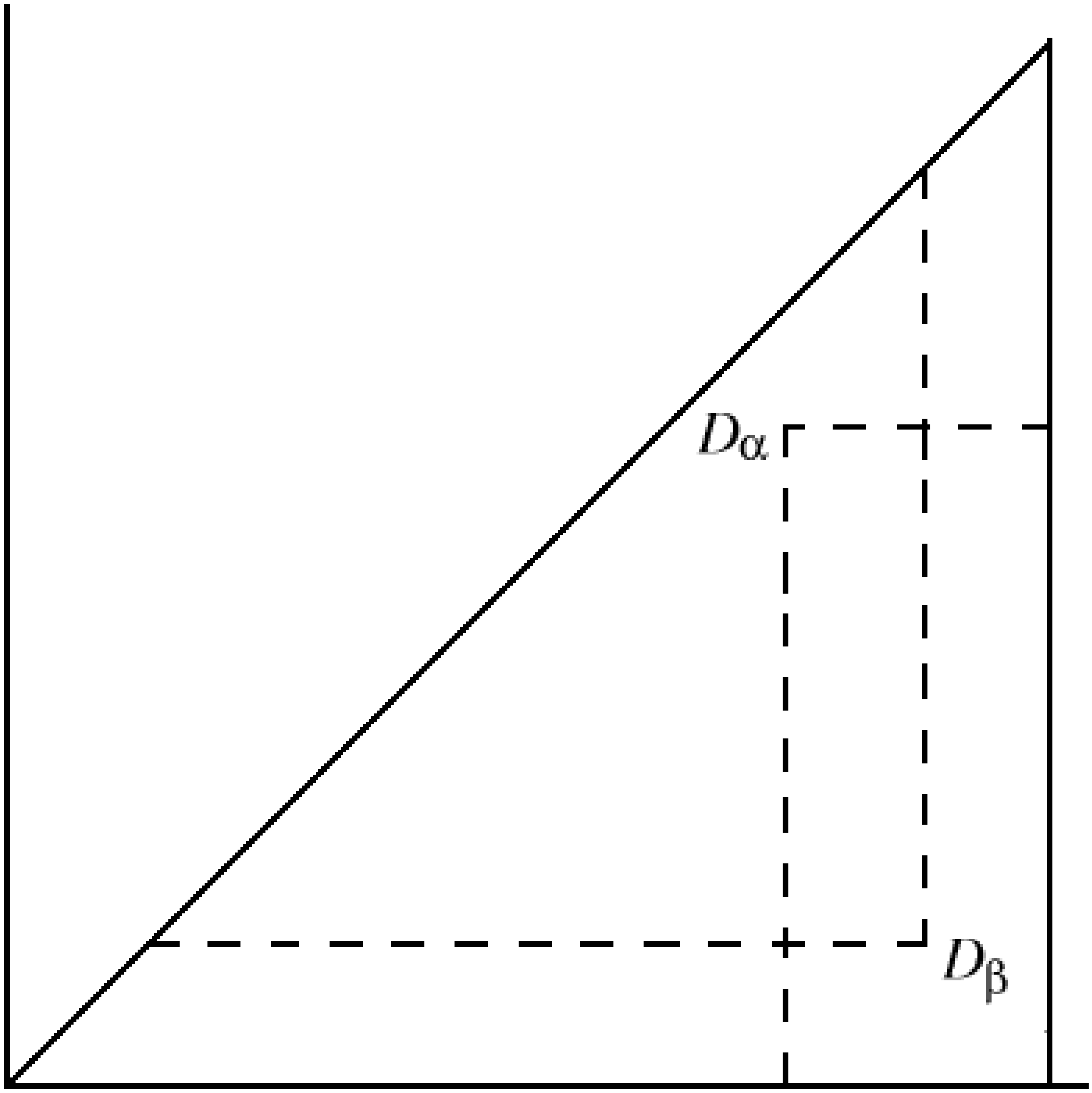
By the above Theorem, if we fix D_α



Conversely, if we fix D_α



Hence the reciprocal positions of D_α and D_β are as follows

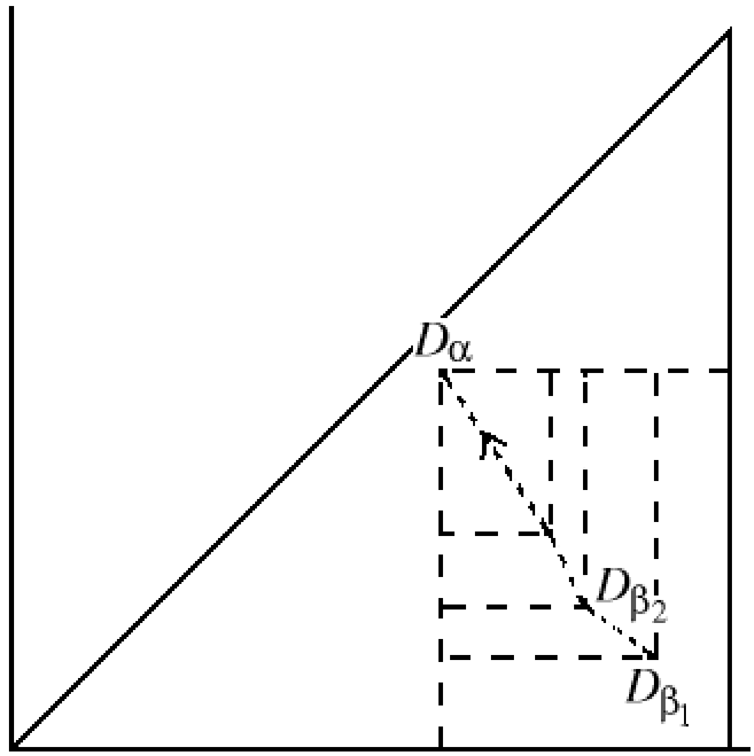


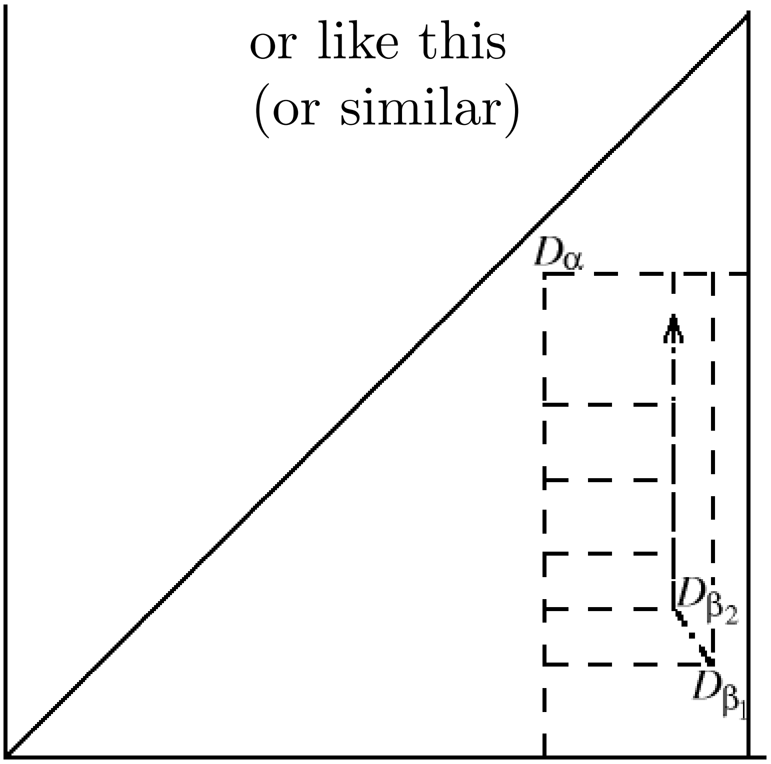
Take now a sequence $(\beta_n)_n$, with

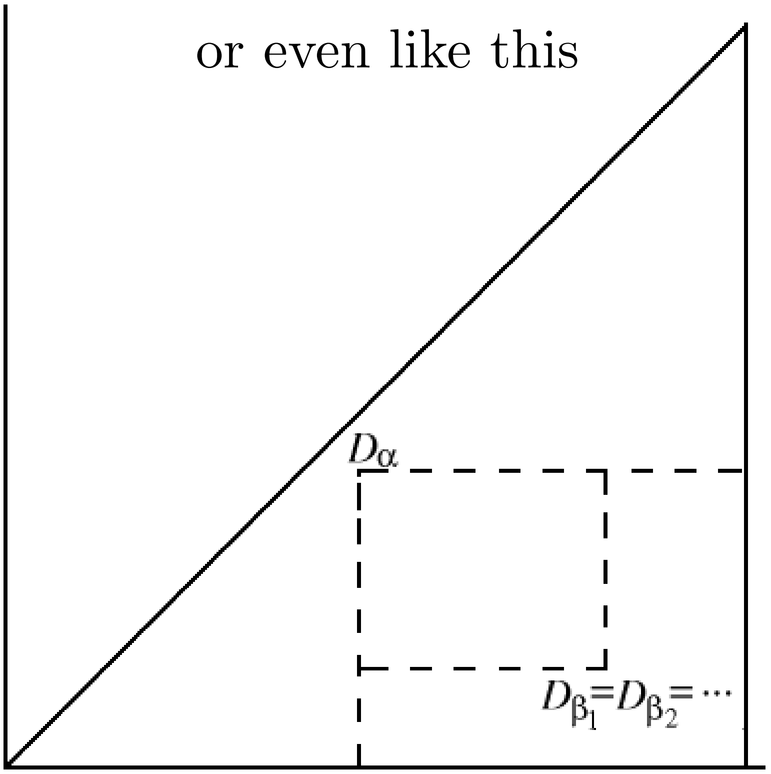
$$\beta_n \to \alpha, \qquad \beta_1 > \beta_2 > \cdots > \beta_n > \cdots \alpha.$$

According to the above remarks, we see that, while β_n moves toα, the rectangles having opposite vertexes in D_α and D_β are "nested "one inside the other.

Hence the situation could be like this




or like this
(or similar)
D_α
D_{β_2}
D_{β_1}


or even like this
D_α
$D_{\beta_1}=D_{\beta_2}=\cdots$

The first picture suggests that

$$(*) \qquad \lim_{\beta \to \alpha} D_\beta = D_\alpha;$$

of course this would mean that both functions $\beta \mapsto \underline{d}_\beta A$ and $\beta \mapsto \underline{d}_\beta A$ are continuous at α.

So, let's investigate whether the above relation (*) is true or not.

It turns out that we have to distinguish two cases:

Case (i): $\alpha > -1$;

Case (ii): $\alpha = -1$.

Case (i).

We have

THEOREM

(i)*Let $\delta > 0$ be fixed. Then, for every set $A \subseteq \mathbb{N}^*$, we have*

$$|\underline{d}_\alpha A - \underline{d}_{\alpha+\delta} A| \leq \frac{2\delta}{\alpha+1};$$

$$|\overline{d}_\alpha A - \overline{d}_{\alpha+\delta} A| \leq \frac{2\delta}{\alpha+1}.$$

(ii) *Let δ be fixed such that $0 < \delta < \alpha + 1$. Then, for every $A \subseteq \mathbb{N}^*$, we have*

$$|\underline{d}_\alpha A - \underline{d}_{\alpha-\delta} A| \leq \frac{2\delta}{\alpha-\delta+1};$$

$$|\overline{d}_\alpha A - \overline{d}_{\alpha-\delta} A| \leq \frac{2\delta}{\alpha-\delta+1}.$$

The above Theorem says that:

- both functions $\beta \mapsto \underline{d}_\beta A$ and $\beta \mapsto \underline{d}_\beta A$ are **lipschitzian**, hence **continuous**, at any $\alpha > -1$;
- they are **uniformly continuous** in any closed half line $[-1+\epsilon, +\infty)$.

And this settles down case (i).

Remark

The Theorem is **uniform** in $A \subseteq \mathbb{N}^*$.

BUT the Lipschitz constant

$$\frac{1}{\alpha+1}$$

goes to ∞ as α goes to- 1!

This compels us to perform different calculations in

Case (ii).

We assume that $A \subseteq \mathbb{N}^*$ is neither finite nor cofinite and represent it as

$$A = \mathbb{N}^* \cap (\cup_{n\geq 1}]a_n, b_n]),$$

where $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ verify

$$a_n < b_n < a_{n+1} \ \forall n.$$

Set

$$C_n = \log b_n - \log a_n;$$

$$B_n = \log b_n - \log b_{n-1};$$

$$A_n = \log a_n - \log a_{n-1}.$$

assume that $(B_n)_{n\geq 1}$ is bounded (assumption $\mathcal{H}$). (Hence $(A_n)_{n\geq 1}$ and $(C_n)_{n\geq 1}$ are bounded as well).

We have

THEOREM 1

In addition to assumption $\mathcal{H}$, suppose that

$$L \doteq \liminf_{n\to\infty} C_n > 0.$$

Then we have

$$\lim_{\alpha\to -1^+} \overline{d}_\alpha A = \overline{d}_{-1} A;$$

$$\lim_{\alpha\to -1^+} \underline{d}_\alpha A = \underline{d}_{-1} A.$$

Assumption $\mathcal{H}$ cannot be dropped; we have in fact:

EXAMPLE.

Take $A = \mathbb{N}^* \cap (\cup_n]a_n, b_n])$, with

$$a_n = n((n-1)!)^2, \qquad b_n = (n!)^2.$$

Assumption $\mathcal{H}$ is not verified; in fact

$$C_n = \log(n!)^2 - \log n((n-1)!)^2 = \log n$$

is not bounded!

And it is easy to verify that

$$\underline{d}_{-1}(A) = \overline{d}_{-1}(A) = d_{-1}(A) = \tfrac{1}{2},$$

while, for every $\alpha > -1$,

$$\underline{d}_\alpha(A) = 0; \qquad \overline{d}_\alpha(A) = 1 :$$

neither function $\alpha \mapsto \underline{d}_\alpha(A)$, $\alpha \mapsto \overline{d}_\alpha(A)$ is continuous in -1!

Theorem 1 covers the rather relevant case of sets such the set E_r of numbers beginning by a fixed digit r ($r \in \{1, 2, \ldots, 9\}$):

$$E_r = \mathbb{N}^* \cap (\cup_n]r10^n - 1, (r+1)10^n - 1]),$$

but is not useful for instance for the set of even numbers (or the set of multiples of any other integer, of course). In fact in this case we have $a_n = 2n - 1$, $b_n = 2n$ and

$$\liminf_n (\log b_n - \log a_n) = 0.$$

Observe that in this case

$$\lim_n \frac{\log b_n - \log a_n}{\log b_n - \log b_{n-1}} = \lim_n \frac{C_n}{B_n}$$

exists ($= 1/2$).

In fact, for $A = \mathbb{N}^* \cap (\cup_n]a_n, b_n])$ the following result holds:

Theorem 2.

Let assumption $\mathcal{H}$ hold and suppose that the limit

$$\lim_{n\to\infty} \frac{C_n}{B_n}$$

exists and is equal to L. Then

(i) *A possesses logarithmic density*

$$d_{-1}(A) = L;$$

(ii) *we have, for $\alpha_0 > \alpha > -1$,*

$$\limsup_{n\to\infty} \left| \frac{\sum_{k=1}^{n}(b_k^{1+\alpha} - a_k^{1+\alpha})}{\sum_{k=1}^{n}(b_k^{1+\alpha} - b_{k-1}^{1+\alpha})} - \frac{\sum_{k=1}^{n} C_k}{\sum_{k=1}^{n} B_k} \right|$$
$$\leq const(1+\alpha).$$

As a consequence we get

$$\lim_{\alpha\to -1^+} \underline{d}_\alpha(A) = \lim_{\alpha\to -1^+} \overline{d}_\alpha(A) = d_{-1}(A).$$

Remark

In case (ii), the results are not uniform in A!

References

[H-R] **H. Halberstam- K. F. Roth**, *Sequences*, Oxford University Press (1966).