Applications of statistical convergence and $\mathcal{I}$-convergence in the theory of numbers.

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Admissible ideal – $\mathcal{I} \subseteq 2^{\mathbb{N}}$

- *contains singletons*: $\{n\} \in \mathcal{I}$,
- *hereditary*: $A \in \mathcal{I}, B \subseteq A \Rightarrow B \in \mathcal{I}$,
- *additive*: $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
- *proper*: $\mathbb{N} \notin \mathcal{I}$.

Examples:

- $\mathcal{I}_f = \{A;\ |A| < +\infty\}$,
- $\mathcal{I}_d = \{A;\ d(A) = 0\}$,
- $\mathcal{I}_c = \{A;\ \sum_{a\in A} 1/a < +\infty\}$,
- $\mathcal{I}_c^q = \{A;\ \sum_{a\in A} 1/a^q < +\infty\}$, $q \in (0,1)$.

$\mathcal{I}$-**convergence** (Kostyrko, Šalát, Wilczyński)

$$\mathcal{I}-\lim x_n = x \iff \forall \varepsilon > 0 : A_\varepsilon \in \mathcal{I}$$

$$(A_\varepsilon = \{n \in \mathbb{N};\ |x_n - x| \geq \varepsilon\}).$$

- $\mathcal{I} = \mathcal{I}_f$ – classical convergence,
- $\mathcal{I} = \mathcal{I}_d$ – statistical convergence.
- $\mathcal{I}-\lim x_n = x,\ \mathcal{I}-\lim x_n = x' \Rightarrow x = x'$,
- $\mathcal{I}-\lim x_n = x,\ \mathcal{I}-\lim y_n = y \Rightarrow$
  $\mathcal{I}-\lim x_n + y_n = x + y,\ \mathcal{I}-\lim x_n y_n = xy$,
- $\mathcal{I} \subseteq \mathcal{I}',\ \mathcal{I}-\lim x_n = x \Rightarrow \mathcal{I}'-\lim x_n = x$.

Olivier (1827)

$$a_n \geq a_{n+1} > 0,\ \sum a_n < +\infty \Rightarrow \lim n a_n = 0.$$

Definition $\mathcal{I} \in \mathcal{S}(T)$ iff

$$a_n > 0,\ \sum a_n < +\infty \Rightarrow \mathcal{I} - \lim n a_n = 0.$$

Facts

$$\mathcal{I}_f \notin \mathcal{S}(T),$$

$$\mathcal{I}_d \in \mathcal{S}(T).$$

Šalát, Toma (2003)

$$\mathcal{I} \in \mathcal{S}(T) \iff \mathcal{I}_c \subseteq \mathcal{I}.$$

$$\mathcal{I}_f \subset \mathcal{I}_c^q \subset \mathcal{I}_c \subset \mathcal{I}_d,$$

$$q < q' \Rightarrow \mathcal{I}_c^q \subset \mathcal{I}_c^{q'}.$$

Definition: Šalát (1994) $n \in \mathbb{N}$, p-prime

$$a_p(n): \ p^{a_p(n)} || n.$$

Definition: Mycielski (1951) $n \in \mathbb{N}$, $n > 1$

$$\gamma(n): \ n = a_1^{b_1} = a_2^{b_2} = \cdots = a_{\gamma(n)}^{b_{\gamma(n)}}$$

$(a_i, b_i \in \mathbb{N}$, all representations$)$,

$$\tau(n): \ \tau(n) = b_1 + b_2 + \cdots + b_{\gamma(n)}.$$

Known results

$$\mathcal{I}_d\text{–}\lim \frac{a_p(n) \log p}{\log n} = 0, \ \mathcal{I}_f\text{–}\lim \frac{a_p(n) \log p}{\log n} \neq 0,$$

$$\mathcal{I}_d - \lim \gamma(n) = 1, \ \mathcal{I}_f - \lim \gamma(n) \neq 1,$$

$$\mathcal{I}_d - \lim \tau(n) = 1, \ \mathcal{I}_f - \lim \tau(n) \neq 1,$$

Z. Fehér, B. László, M.M., T. Šalát (2004)

$$\mathcal{I}_c - \lim \frac{a_p(n) \log p}{\log n} = 0,$$

$$\mathcal{I}_c^q - \lim \frac{a_p(n) \log p}{\log n} \neq 0,$$

$$q > \frac{1}{2} \Rightarrow \mathcal{I}_c^q - \lim \gamma(n) = 1,$$

$$q \leq \frac{1}{2} \Rightarrow \mathcal{I}_c^q - \lim \gamma(n) \neq 1,$$

$$q > \frac{1}{2} \Rightarrow \mathcal{I}_c^q - \lim \tau(n) = 1,$$

$$q \leq \frac{1}{2} \Rightarrow \mathcal{I}_c^q - \lim \tau(n) \neq 1.$$