On density and multiplicative structure of sets of generalized integers

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$A \subset \mathbb{N}, \qquad N_A(x) = \sum_{\substack{a_i \in A \\ a_i \le x}} 1, \qquad x > 1$

• *asymptotic density*:

$$\underline{d}(A) = \liminf_{x\to\infty} \frac{N_A(x)}{x}, \qquad \overline{d}(A) = \limsup_{x\to\infty} \frac{N_A(x)}{x}$$

• *logarithmic density*:

$$\underline{\ell}(A) = \liminf_{x\to\infty} \frac{1}{\log x} \sum_{\substack{a_i \in A \\ a_i \le x}} \frac{1}{a_i}, \ \overline{\ell}(A) = \limsup_{x\to\infty} \frac{1}{\log x} \sum_{\substack{a_i \in A \\ a_i \le x}} \frac{1}{a_i}$$

• *Schnirelmann density*:
$$h(A) = \inf_x \left\{ \frac{N_A(x)}{x} \right\}$$

$$\boxed{0 \le \underline{d}(A) \le \underline{\ell}(A) \le \overline{\ell}(A) \le \overline{d}(A)}$$

<u>Common features</u>:

a. density is a non–negative real number
b. finite sets have zero density
c. if $A \subset B \subset \mathbb{N}$ then density of A does not exceed density of B

Arithmetical semigroups

- $(\mathbb{G}, .)$ free commutative semigroup with identity element $1_{\mathbb{G}}$
- $P_{\mathbb{G}} \leq \infty$ the set of generators (the so–called **primes**)
- **norm** $|\cdot| : \mathbb{G} \to \mathbb{R}$:
  - $\star$ $|1_{\mathbb{G}}| = 1, |a| > 1$ for all $a \in \mathbb{G}$,
  - $\star$ $|ab| = |a|.|b|$ for all $a, n \in \mathbb{G}$,
  - $\star$ $N_{\mathbb{G}}(x) = \sum_{\substack{|a| \leq x \\ a \in \mathbb{G}}} 1 < \infty$ for each real x.

Axiom A: *There exists positive constants A and δ and a constant η with $0 \leq \eta < \delta$, such that*

$$N_{\mathbb{G}}(x) = Ax^{\delta} + \mathcal{O}(x^{\eta})$$

ζ–function of $\mathbb{G}$: $\zeta_{\mathbb{G}}(s) = \sum\limits_{a \in \mathbb{G}} \dfrac{1}{|a|^s}$

Lemma. *Let $\mathbb{G}$ be an arithmetical semigroup satisfying Axiom A. Then $\sum_{a \in \mathbb{G}} |a|^{-z}$ is absolutely convergent for $\Re(z) > \delta$, and divergent for $\Re(z) \leq \delta$. Moreover*

$$\sum_{|a| \leq x} |a|^{-\delta} = \delta A \log x + \gamma_{\mathbb{G}} + \mathcal{O}(x^{\eta - \delta}).$$

Example 1. $\mathbb{G} = \mathbb{N}$, the set of positive integers where

$$N_{\mathbb{N}}(x) = x + \mathcal{O}(1).$$

Example 2. $\mathbb{G} = G_K$, the semigroup of all non–zero integral ideals in a given algebraic number field K of degree $n = [K : \mathbb{Q}]$ over rationals $\mathbb{Q}$ with the usual norm function $|\mathfrak{a}| = \text{card}(O_K/\mathfrak{a})$. Then

$$N_K(x) = A_K x + \mathcal{O}\left(x^{\frac{n-1}{n+1}}\right),$$

where A_K can be explicitly given.

Example 3. $\mathbb{G} = \mathbb{A}$ the category of all finite Abelian groups with the usual direct product operation and the norm $|H| = \text{card}(H)$. Fundamental Theorem on finite Abelian groups shows that $\mathbb{A}$ is free and that the generators are the cyclic groups of prime–power order. The fact that this arithmetical semigroup satisfies Axiom A follows from an older result of Erdős and Szekeres that

$$N_{\mathcal{A}}(x) = \alpha x + \mathcal{O}\left(\sqrt{x}\right),$$

where $\alpha = \prod_{j=1}^{\infty} \zeta_{\mathbb{N}}(js)$ with $\zeta_{\mathbb{N}} = \zeta$ denoting the classical Riemann zeta function.

$\delta \in (-\infty, +\infty)$

***δ*–regularly varying function** function $F(x)$ defined and measurable on $\langle 0, \infty)$:

$$\lim_{x\to\infty} \frac{F(\lambda x)}{F(x)} = \lambda^{\delta} \quad \text{if for every } \lambda > 0$$

If $F(x) = x^{\delta} L(x)$, then $L(x)$ is **slowly oscillating** (i.e. $\delta = 0$).

Arithmetical semigroup $\mathbb{G}$ will be called ***δ*–regular** if the counting function $N_{\mathbb{G}}(x)$ is δ–regularly varying function.

Lemma. *Let $\mathbb{G}$ be a δ–regular semigroup. Then $\sum_{a\in\mathbb{G}} |a|^{-z}$ is convergent for all $\Re(z) > \delta$ and divergent for all $\Re(z) < \delta$.*

Wegmann (1966): If $N_{\mathbb{G}}(x) \sim \dfrac{x}{\log^2 x}$ (i.e. $\mathbb{G}$ is 1–regular), then $\sum_{a\in\mathbb{G}} |a|^{-1} < \infty$.

$\mathfrak{m} \; : \; \mathbb{G} \to \mathbb{R}$, $\qquad \mathcal{C} \subset \mathbb{G}$: $\chi_{\mathcal{C}}$ indicator of C

$N_{\mathcal{C}}(\mathfrak{m}, x) = \sum_{|a| \le x} \mathfrak{m}(a) \chi_{\mathcal{C}}(a)$

$N'_{\mathcal{C}}(\mathfrak{m}, x) = \sum_{|a| = x} \mathfrak{m}(a) \chi_{\mathcal{C}}(a)$

$$\sigma_x(\mathcal{C}, \mathfrak{m}) = \frac{\sum_{|a| \le x} \mathfrak{m}(a) \chi_{\mathcal{C}}(a)}{\sum_{|a| \le x} \mathfrak{m}(a)} = \frac{N_{\mathcal{C}}(\mathfrak{m}, x)}{N_{\mathbb{G}}(\mathfrak{m}, x)}$$

lower $\mathfrak{m}$*–density*: $\qquad \underline{\sigma}(\mathcal{C}, \mathfrak{m}) = \liminf_{x \to \infty} \sigma_x(\mathcal{C}, \mathfrak{m})$

upper $\mathfrak{m}$*–density*: $\qquad \overline{\sigma}(\mathcal{C}, \mathfrak{m}) = \limsup_{x \to \infty} \sigma_x(\mathcal{C}, \mathfrak{m})$

A. $\mathfrak{m}$ is non–negative, i.e. $\mathfrak{m}(a) \ge 0$ for every $a \in \mathbb{G}$
B. $\sum\limits_{a \in \mathbb{G}} \mathfrak{m}(a)$ diverges

A implies **a**

B implies **b**

B implies the summation method is regular

Knopp's theorem on convergence kernel:

Theorem. *Let* $\mathfrak{m}$ *and* $\mathfrak{s}$ *be two positive functions defined on an arithmetical semigroup* $\mathbb{G}$ *such that*

(i) *the series* $\sum_{a\in\mathbb{G}} \mathfrak{s}(a)$ *diverges*

(ii) $$\lim_{x\to\infty} \frac{N'_{\mathbb{G}}(\mathfrak{m},x)}{N_{\mathbb{G}}(\mathfrak{m},x)} = \lim_{x\to\infty} \frac{\sum_{|a|=x} \mathfrak{m}(a)}{\sum_{|a|\le x} \mathfrak{m}(a)} = 0$$

$$\lim_{x\to\infty} \frac{N'_{\mathbb{G}}(\mathfrak{s},x)}{N_{\mathbb{G}}(\mathfrak{s},x)} = \lim_{x\to\infty} \frac{\sum_{|a|=x} \mathfrak{s}(a)}{\sum_{|a|\le x} \mathfrak{s}(a)} = 0$$

(iii) *if* $a_1, a_2 \in \mathbb{G}$ *be such that* $|a_1| \le |a_2|$ *then*

$$\frac{\mathfrak{m}(a_2)}{\mathfrak{m}(a_1)} \ge \frac{\mathfrak{s}(a_2)}{\mathfrak{s}(a_1)}.$$

Then

$$\underline{\sigma}(C,\mathfrak{m}) \le \underline{\sigma}(C,\mathfrak{s}) \le \overline{\sigma}(C,\mathfrak{s}) \le \overline{\sigma}(C,\mathfrak{m})$$

for every $C \subset \mathbb{G}$.

If $N_{\mathbb{G}}(\mathfrak{m}, x)$ is δ–regular, (ii) is superfluous:

$\sum_{|a|\leq x} \mathfrak{m}(n) = x^{\delta} L(x)$, and $0 < \alpha < 1$ arbitrary

$$0 \leq \frac{\sum_{|a|=x} \mathfrak{m}(n)}{\sum_{|a|\leq x} \mathfrak{m}(n)} \leq \frac{\sum_{\alpha x<|a|\leq x} \mathfrak{m}(n)}{\sum_{|a|\leq x} \mathfrak{m}(n)}$$

$$= \frac{x^{\delta} L(x) - \alpha^{\delta} x^{\delta} L(\alpha x)}{x^{\delta} L(x)} \to 1 - \alpha^{\delta}$$

M. to every $a \in \mathbb{G}$ there exists a positive real number $\widehat{\mathfrak{m}}(a)$, $\widehat{\mathfrak{m}} < 1$ such that for every subset $\mathcal{C} \subset \mathbb{G}$ having the $\mathfrak{m}$–density $\sigma(\mathcal{C}, \mathfrak{m})$ the set $a\mathcal{C} = \{ac : c \in \mathbb{G}\}$ has also the $\mathfrak{m}$–density and

$$\sigma(a\mathcal{C}, \mathfrak{m}) = \widehat{\mathfrak{m}}(a)\sigma(\mathcal{C}, \mathfrak{m}).$$

- if $\mathfrak{m}(a) = 1$ for every $a \in \mathbb{G}$ then $\widehat{\mathfrak{m}}(a) = \dfrac{1}{a}$
- if $\mathfrak{m}$ is completely multiplicative and $N_{\mathbb{G}}(\mathfrak{m}, x) = x^{\Delta} L(x)$, where $L(x)$ is slowly oscillating, then

$$\widehat{\mathfrak{m}}(a) = \mathfrak{m}(a)|a|^{-\Delta}.$$

Question:

Under which conditions does $\widehat{\mathfrak{m}}$ fulfil the conditions **A** and **B**?

Theorem. *Let $\mathbb{G}$ be an arithmetical semigroup. Let $\mathfrak{m}$ satisfy conditions* **A**, **B**, *and* **M**. *Let in the case, when*

$$\sum_{p \in P_{\mathbb{G}}} \widehat{\mathfrak{m}}(p) < \infty, \qquad (*)$$

we have uniformly in x and $p \in P_{\mathbb{G}}$

$$\sigma_{p\mathbb{G}}(\mathfrak{m}, x) = \mathcal{O}(\widehat{\mathfrak{m}}(p)) \qquad (**)$$

Then $\widehat{\mathfrak{m}}$ fulfils conditions **A** *and* **B**.

Note: $(**)$ cannot be omitted if $(*)$ holds!

Take Wegmann's $\mathbb{G}$ with asymptotic density. Then $(*)$ holds ($\widehat{\mathfrak{m}}(p) = |p|^{-1}$), while it can be shown that the *finite* set
$\mathbb{G}\langle P_{\mathbb{G}}\rangle = \{a \in \mathbb{G} \ : \ p \nmid a$ for every $p \in P_{\mathbb{G}}\} = \{1_{\mathbb{G}}\}$
has *non–zero density*

$$\prod_{p \in P_{\mathbb{G}}} (1 - |p|^{-1}).$$

Lemma. *If the arithmetical semigroup $\mathbb{G}$ satisfies Axiom A then the series*

$$\sum_{p \in P_{\mathbb{G}}} |p|^{-\delta}$$

diverges.

We have

- $\widehat{\mathfrak{m}}$ is *completely multiplicative* $(ab\mathbb{G} = a(b\mathbb{G}))$,
- $\lim\limits_{|a| \to \infty} \widehat{\mathfrak{m}}(a) = 0$ (if $\lim_{|p| \to \infty} \widehat{\mathfrak{m}}(p) = 0$).

Therefore

- $a \mapsto \dfrac{1}{\widehat{\mathfrak{m}}(a)}$ is a norm on $\mathbb{G}$
- $\widehat{\zeta}_{\mathbb{G}}(z) = \sum\limits_{a \in \mathbb{G}} \left(\dfrac{1}{\widehat{\mathfrak{m}}(a)} \right)^{-z} = \sum\limits_{a \in \mathbb{G}} \widehat{\mathfrak{m}}(a)^z$

B says that $\widehat{\zeta}_{\mathbb{G}}(z)$ has a pole at $z = 1$

if $\mathfrak{m}(a) = 1$ for every $a \in \mathbb{G}$ and $\mathbb{G}$ is δ–regular, then $\widehat{\zeta}_{\mathbb{G}}(z) = \zeta_{\mathbb{G}}(s\delta)$, and therefore **B** holds if $\mathbb{G}$ satisfies Axiom A.

slowly oscillating function $L(x)$ is called **good** if $\lim_{x\to\infty} Z(x) = \infty$, where $Z(x) = \int_1^x L(y)y^{-1}\mathrm{d}y$.

if $\liminf_{x\to\infty} L(x) > 0$, then L is good

Theorem. *Let $\mathfrak{m}$ be a completely multiplicative function defined on an arithmetical semigroup $\mathbb{G}$. Let*

$$N_{\mathbb{G}}(\mathfrak{m}, x) = \sum_{|a|\le x} \mathfrak{m}(a) = x^{\delta} L(x),$$

with $L(x)$ a good slowly oscillating function. Then

$$0 \le \underline{\sigma}(\mathcal{C}, \mathfrak{m}) \le \underline{\sigma}(\mathcal{C}, \widehat{\mathfrak{m}}) \le \overline{\sigma}(\mathcal{C}, \widehat{\mathfrak{m}}) \le \overline{\sigma}(\mathcal{C}, \mathfrak{m}) \le 1$$

for every $\mathcal{C} \subset \mathbb{G}$.

Corollary. *Let $\mathfrak{m}$ be a positive completely multiplicative function defined on the arithmetical semigroup $\mathbb{G}$ such that*

$$\sum_{|a|\le x} \mathfrak{m}(a) = L(x),$$

where L is a good slowly oscillating function, then

$$\underline{\sigma}(\mathcal{C}, \mathfrak{m}) = \underline{\sigma}(\mathcal{C}, \widehat{\mathfrak{m}}), \quad \text{and} \quad \overline{\sigma}(\mathcal{C}, \widehat{\mathfrak{m}}) = \overline{\sigma}(\mathcal{C}, \mathfrak{m})$$

for every $\mathcal{C} \subset \mathbb{G}$.

Corollary. *Let $\mathfrak{m}$ be a completely multiplicative function defined on an arithmetical semigroup $\mathbb{G}$. Let*

$$N_{\mathbb{G}}(\mathfrak{m}, x) = \sum_{|a| \leq x} \mathfrak{m}(a) = x^{\delta} L(x),$$

with $L(x)$ a good slowly oscillating function. Then

$$\widehat{\widehat{\mathfrak{m}}} = \widehat{\mathfrak{m}}.$$

Corollary. *Let the arithmetical semigroup $\mathbb{G}$ satisfies Axiom A. Let $\mathfrak{m}$ be a positive function defined on $\mathbb{G}$ such that*

- *the series $\sum_{a \in \mathbb{G}} \mathfrak{m}(a)$ diverges,*
- $\displaystyle \lim_{x \to \infty} \frac{N'_{\mathbb{G}}(\mathfrak{m}, x)}{N_{\mathbb{G}}(\mathfrak{m}, x)} = \lim_{x \to \infty} \frac{\sum_{|a| = x} \mathfrak{m}(a)}{\sum_{|a| \leq x} \mathfrak{m}(a)} = 0,$
- $|a_1| \leq |a_2| \Rightarrow \mathfrak{m}(a_1) \leq \mathfrak{m}(a_2)$

then the lower and upper $\mathfrak{m}$–density coincides with the lower and upper logarithmic density.

Theorem. *Let* $\mathfrak{m} : \mathbb{G} \to \mathbb{R}^+$ *be a completely multiplicative function such that*

$$N_{\mathbb{G}}(\mathfrak{m}, x) = Bx^{\Delta} + \mathcal{O}(x^{\Theta}), \quad 0 \le \Theta < \Delta$$

as $x \to \infty$, *then*

$$N_{\mathbb{G}}(\widehat{\mathfrak{m}}, x) = \Delta B \log x + \psi_m + \mathcal{O}(x^{\Theta-\Delta})$$

with a suitable ψ_m.

lower $\mathfrak{m}$–density:

$$\underline{\sigma}_{\mathbb{G}}(\mathcal{C}, \mathfrak{m}) = \liminf_{x\to\infty} \frac{N_{\mathcal{C}}(\mathfrak{m}, x)}{N_{\mathbb{G}}(\mathfrak{m}, x)} = \liminf_{x\to\infty} \frac{\sum\limits_{a\in\mathcal{C}, |a|\le x} \mathfrak{m}(a)}{Bx^{\Delta}}$$

upper $\mathfrak{m}$–density:

$$\overline{\sigma}_{\mathbb{G}}(\mathcal{C}, \mathfrak{m}) = \limsup_{x\to\infty} \frac{N_{\mathcal{C}}(\mathfrak{m}, x)}{N_{\mathbb{G}}(\mathfrak{m}, x)} = \limsup_{x\to\infty} \frac{\sum\limits_{a\in\mathcal{C}, |a|\le x} \mathfrak{m}(a)}{Bx^{\Delta}}$$

lower $\widehat{\mathfrak{m}}$–density:

$$\underline{d}_{\mathbb{G}}(\mathcal{C}) = \liminf_{x\to\infty} \frac{N_{\mathcal{C}}(\widehat{\mathfrak{m}}, x)}{N_{\mathbb{G}}(\widehat{\mathfrak{m}}, x)} = \liminf_{x\to\infty} \frac{\sum\limits_{a\in\mathcal{C}, |a|\le x} \mathfrak{m}(a)|a|^{-\Delta}}{\Delta B \log x}$$

upper $\widehat{\mathfrak{m}}$–density:

$$\overline{d}_{\mathbb{G}}(\mathcal{C}) = \limsup_{x\to\infty} \frac{N_{\mathcal{C}}(\widehat{\mathfrak{m}}, x)}{N_{\mathbb{G}}(\widehat{\mathfrak{m}}, x)} = \limsup_{x\to\infty} \frac{\sum\limits_{a\in\mathcal{C}, |a|\le x} \mathfrak{m}(a)|a|^{-\Delta}}{\Delta B \log x}$$

if Axiom A holds for $\mathbb{G}$ then $\sum_{|a|\le x} |a|^{-\delta} \sim \delta A \log x$, and

$$N_{\mathbb{G}}(x) \sim A x^{\delta}$$

lower logarithmic density:

$$\underline{\ell}_{\mathbb{G}}(\mathcal{C}) = \liminf_{x\to\infty} \frac{1}{\delta A \log x} \sum_{|a|\le x} |a|^{-\delta}$$

upper logarithmic density:

$$\overline{\ell}_{\mathbb{G}}(\mathcal{C}) = \limsup_{x\to\infty} \frac{1}{\delta A \log x} \sum_{|a|\le x} |a|^{-\delta}$$

lower asymptotic density:

$$\underline{d}_{\mathbb{G}}(\mathcal{C}) = \liminf_{x\to\infty} \frac{N_{\mathcal{C}}(\mathtt{1}, x)}{N_{\mathbb{G}}(x)} = \liminf_{x\to\infty} \frac{\sum_{a\in\mathcal{C}, |a|\le x} 1}{A x^{\delta}}$$

upper asymptotic density:

$$\overline{d}_{\mathbb{G}}(\mathcal{C}) = \limsup_{x\to\infty} \frac{N_{\mathcal{C}}(\mathtt{1}, x)}{N_{\mathbb{G}}(x)} = \limsup_{x\to\infty} \frac{\sum_{a\in\mathcal{C}, |a|\le x} 1}{A x^{\delta}}$$

Corollary. *Let* $\mathfrak{m} : \mathbb{G} \to \mathbb{R}^+$ *be a completely multiplicative function such that*

$$N_{\mathbb{G}}(\mathfrak{m}, x) = Bx^{\Delta} + \mathcal{O}(x^{\Theta}), \quad 0 \leq \Theta < \Delta$$

as $x \to \infty$, *then*

$$0 \leq \underline{\sigma}_{\mathbb{G}}(\mathcal{C}, \mathfrak{m}) \leq \underline{\sigma}_{\mathbb{G}}(\mathcal{C}, \widehat{\mathfrak{m}}) \leq \overline{\sigma}_{\mathbb{G}}(\mathcal{C}, \widehat{\mathfrak{m}}) \leq \overline{\sigma}_{\mathbb{G}}(\mathcal{C}, \mathfrak{m}) \leq 1$$

for every $\mathcal{C} \subset \mathbb{G}$.

Corollary. *If the arithmetical semigroup* $\mathbb{G}$ *satisfies Axiom A, then*

$$0 \leq \underline{d}_{\mathbb{G}}(\mathcal{C}) \leq \underline{\ell}_{\mathbb{G}}(\mathcal{C}) \leq \overline{\ell}_{\mathbb{G}}(\mathcal{C}) \leq \overline{d}_{\mathbb{G}}(\mathcal{C}) \leq 1$$

for every $\mathcal{C} \subset \mathbb{G}$.

Thank you!