Problem session notes

We would like to ask the authors of problems to check and complete the formulations and also correct references if they are wrong.

Density set for exponential density (proposed by Georges Grekos)

Definitions and motivation: Properties of the density sets have been studied in [GREVOL]. One can ask similar questions for *exponential density* given by $\overline{\varepsilon}A = \limsup \frac{\log A(n)}{\log n}$ and $\underline{\varepsilon}A = \liminf \frac{\log A(n)}{\log n}$. Let $E(A) = (\overline{\varepsilon}A, \underline{\varepsilon}A)$ be the density point associated with a subset $A \subseteq N$. For any $A \subseteq \mathbb{N}$ we denote $S(A) = \{E(B) : B \subseteq A\}$ and $T(A) = \{E(C) : C \supseteq A\}$. It is known (see thesis [GREKOS]) that $T(A)$ is the trapezium given by the points $(\overline{\varepsilon}A, \underline{\varepsilon}A)$, $(\overline{\varepsilon}A, \overline{\varepsilon}A)$, $(1, 1)$ and $(1, \underline{\varepsilon}A)$ and that the set $S(A)$ is closed and the upper boundary of this set is a continuous function $[0, \overline{\varepsilon}A] \to [0, \underline{\varepsilon}A]$.

Questions:

1. Is the set $S(A)$ convex?
2. Find a characterization of $S(A)$.

Sum sets and Kneser pairs (proposed by Bodo Volkmann)

Definitions and motivation: For subsets $A, B \subseteq \mathbb{N}$ we define the *sum set* $A+B = \{a+b; a \in A \cup \{0\}, b \in B \cup \{0\}\}$. One of the long standing problems of the last century was the so called $(\alpha+\beta)$-hypothesis. It was solved in [MANN], where the inequality $\sigma(A+B) \geq \min\{1, \sigma(A)+\sigma(B)\}$ was proved ($\sigma(A)$ is the Schnirelmann density). Several easy counterexamples show that the same inequality $\underline{d}(A+B) \geq \min\{1, \underline{d}(A)+\underline{d}(B)\}$ doesn't hold for lower asymptotic density. It took more than ten years until the analogous result to this theorem was found by [KNESER]. This result reads as follows:

a) Either $\underline{d}(A+B) \geq \underline{d}(A)+\underline{d}(B)$ or

b) $\underline{d}(A+B) \geq \underline{d}(A^{(k)})+\underline{d}(B^{(k)})-\frac{1}{k}$,

where k is smallest integer such that $A+B$ is (starting from some integer N) union of some residue classes modulo k and $A^{(k)} = \{x;\ x = a \pmod{k}$ for some $a \in A\}$.

Let us call the sets A, B a *Kneser pair* if $\underline{d}(A)+\underline{d}(B) > \underline{d}(A+B)$, and thus b) holds.

Question: Characterize Kneser pairs without using the sum set $A+B$.

Comments: This problem can be related to gap-density. It can be also related to the residue class distribution of the set A, i.e., by intersections of A with residue class modulo each natural number. But no theory of residue class distribution has been developed so far.

Sets which differ from the union of several residue classes modulo some integer k only by a finite set are called *rational sets*. If A and B is a Kneser pair, then $A+B$ is a rational set but the converse doesn't hold in general.

Strong limit points and shift invariant ideals (proposed by Martin Mačaj)

Definitions and motivation: If $\mathbb{Q} = \{r_1, \ldots, r_n, \ldots\}$ is the set of all rational numbers and $P = \{p_1 < p_2 < \ldots < p_n < \ldots\}$ is the set of all primes (in the natural order) then the sequence (x_n) given by

$$x_n = \begin{cases} r_k, & \text{if } n = p_k; \\ 0, & \text{otherwise} \end{cases}$$

has the set of all limit points equal to $\mathbb{R}$. But one can feel that 0 is "more important limit point" of this sequence than other rationals, in fact it is the "most important" one. The following characterization of limit points is known: L is a limit point of (x_n) if and only if $\mathcal{I}\text{-}\lim x_n = L$ for some admissible ideal $\mathcal{I}$.

This motivates the following two definitions. We say that $\mathcal{I}$ is a *shift invariant ideal* if it is admissible and $\mathcal{I}\text{-}\lim x_n = \mathcal{I}\text{-}\lim x_{n+1}$ whenever at least one of the $\mathcal{I}$-limits exists. Equivalently we can define a shift invariant ideal as an admissible ideal which fulfills the condition $A \in \mathcal{I} \Leftrightarrow A + 1 \in \mathcal{I}$.

We say that L is a strong limit point of (x_n) if $\mathcal{I}\text{-}\lim x_n = L$ for some shift invariant ideal $\mathcal{I}$.

Questions:

1. Is it possible to have 2 or more strong limit points?
2. Find a property $\mathcal{P}$ such that the following proposition holds: l is a strong limit point of a sequence $(x_n)_{n=1}^{\infty}$ if and only if $\forall \varepsilon > 0\ \mathcal{P}(A_\varepsilon)$. [We denote $A_\varepsilon = \{x \in \mathbb{R}; \|x - l\| < \varepsilon\}$.]

Partial answer: (Martin Sleziak) The answer to the first question is affirmative. Let us put $B = \bigcup_{k \in \mathbb{N}_0} \{2^{2k}, \ldots, 2^{2k+1} - 1\}$, i.e. B is the set of all positive integers such that their binary representation has odd number of digits. We denote by $\mathcal{I}_1$ the smallest shift invariant ideal containing B and by $\mathcal{I}_2$ the smallest shift invariant ideal containing $\mathbb{N} \setminus B$. These ideals are proper and if we take a sequence $x_n = \chi_B(n)$ then we get $\mathcal{I}_1 - \lim x_n = 1$ and $\mathcal{I}_2 - \lim x_n = 0$. So the $\mathcal{I}$-limit for shift invariant ideals is not necessarily unique and this is an example of a sequence which has at least 2 strong limit points.

So it is natural to ask if we could improve the original definition of strong limit point in a "reasonable" way. We can also investigate set of strong limit points of a sequence of real numbers.

Uniform distributed subsequences (proposed by Ladislav Mišík)

Definitions and motivation: Let (x_n) be a given sequence and let us define $\mathcal{U} = \{A \subseteq \mathbb{N}; (x_n)_{n \in A} \text{ is uniformly distributed} \mod 1\}$.

Question: What can be said about the set $\mathcal{U}$? (Hausdorff dimension, Baire class, porosity of the image of this set under the mapping $A \mapsto \sum_{a \in A} 2^{-a}$ into the interval $[0, 1)$ and other similar questions.)

Comments: This question could be related to some characterizations of deterministic numbers obtained in [RAUZY]. Probably the answer could be that A is a characteristic set of a deterministic number.

Density measures (proposed by Mark Fey)
Definitions and motivation: We denote by dA the asymptotic density of a set $A \subseteq \mathbb{N}$. Let μ be a *density measure:* a finitely additive measure on $2^{\mathbb{N}}$ which extends d, i.e., $\mu(A) = d(A)$ if $d(A)$ exists.

Fix $A \subseteq \mathbb{N}$ and denote $s(A) = \sup\{dB : B \subseteq A; dB \text{ exists }\}$, $t(A) = \inf\{dB : B \supseteq A; dB \text{ exists}\}$. Clearly $s(A) \leq \mu(A) \leq t(A)$.
Question: For a fixed A and all possible density measures what are all possible values of $\mu(A)$?
Comments: This can be related to some results of [BUCK] or [van DOUWEN].

Density sets (proposed by Professor Stylianus C. Pichorides, presented by Georges Grekos)
Definitions and motivation: Suppose that we are given two regions $S_1, S_2 \subseteq T$, where T is the triangle (0,0), (1,0), (1,1). (By region we always mean a closed set of points lying between an initial segment on the x-axis and some increasing continuous function.) If $S_1 \subseteq S_2$ are both convex regions then there exist $A_1 \subseteq A_2 \subseteq \mathbb{N}$ such that $\mathcal{S}(A_i) = S_i$ for $i = 1, 2$. (Symbol $\mathcal{S}(A)$ denotes the density set – $\mathcal{S}(A) = \{(\underline{d}B, \overline{d}B);\ B \subseteq A\}$.)
Question: Let S be a convex region such that $S \subseteq T$. Does there exist a set $A \subseteq \mathbb{N}$ such that for each convex region $R \subseteq S$ one can find $B = B(R) \subseteq A$ with $\mathcal{S}(B) = R$?

Sum of the binary digits (proposed by A. O. Gelfond, presented by Michel Mendés-France) In his article [GELFOND] A.O.Gelfond asks the following question (among many others). Let $S(n)$ be the sum of the binary digits of the positive integer n. Is it true that the density of those n for which $S(n^2)$ is even, equals 1/2? (Of course the problem holds for any basis and any modulus $m > 1$, and any polynomial $P(n)$ of degree > 1).

The sequence $S(n^2)$ is obviously quite complex (however one understands "complexity"). In particular, J. P. Allouche proved that it is not an automatic sequence. See his paper [ALLOUCHE]. But that does not solve Gelfond's problem.

References

[ALLOUCHE] J. P. Allouche: *Somme des chiffres et transcendance*, Bulletin Soc. Math. France **110** (1982), 279–285. (French)

[BUCK] R.C. Buck: *The measure theoretic approach to density,* Amer. J. Math. **68** (1946), 560–580.

[van DOUWEN] E. K. van Douwen: *Finitely additive measures on* $\mathbb{N}$*,* Topology and its Applications **47** (1992), 223–268.

[GELFOND] A. O. Gelfond: *Sur les nombres qui ont des propriétés additives et multiplicatives donnes*, Acta Arithm. **13** (1968), 259–265. (French)

[GREKOS] G. Grekos: *Sur le répartition des densités des sous-suites d'une suite donnée,* Université de Paris 6, 1976. (French)

[GREVOL] G. Grekos, B. Volkmann: *On densities and gaps,* J. Number Theory **26** no. 2 (1987), 129–148. (English. French, German summary)

[KNESER] M. Kneser: *Abshätzung der asymptotischen Dichte von Summenmengen,* Math. Z. **58** (1953), 459–484. (German)

[KNESER2] M. Kneser: *Summenmengen in lokal kompakten Abelschen Gruppen,* Math. Z. **66** (1956), 88–110. (German)

[MANN] H. B. Mann: *A proof of the fundamental theorem on the density of sums of sets of positive integers,* Ann. Math. **(2) 43** (1942), 523-527.

[MANN2] H. B. Mann: *Addition theorems: The addition theorems of group theory and number theory.* Interscience tracts, Interscience, New York, 1965.

[RAUZY] G. Rauzy: *Propriétés statistiques des suites arithmétiques,* Le Mathmaticien, No. 15. Collection SUP. Presses Universitaires de France, Paris, 1976. (French)