$\mathcal{I}$-continuity in topological spaces

1. Definitions

Definition 1.1. A family $\mathcal{I}$ of subsets of $\mathbb{N}$ is an *ideal* in $\mathbb{N}$ if

(1) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,

(2) $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$.

Let us call an ideal $\mathcal{I}$ in $\mathbb{N}$ *proper* if $\mathbb{N} \notin \mathcal{I}$. $\mathcal{I}$ is *admissible* if $\mathcal{I}$ is proper and it contains every singleton. If $\mathcal{I}$ is an ideal in $\mathbb{N}$ then $\mathcal{F}(\mathcal{I}) = \{A : \mathbb{N} \setminus A \in \mathcal{I}\}$ is the *filter associated with the ideal* $\mathcal{I}$.

Definition 1.2. Let $\mathcal{I}$ be an ideal in $\mathbb{N}$. A sequence $(x_n)_{n=1}^{\infty}$ in a topological space X is said to be $\mathcal{I}$-*convergent* to a point $x \in X$ if

$$A(U) = \{n : x_n \notin U\} \in \mathcal{I}$$

holds for each open neighborhood U of x. We denote it by $\mathcal{I}\text{-}\lim x_n = x$.

$\mathcal{I}_f$ = Fréchet ideal (finite subsets of $\mathbb{N}$)
$\mathcal{I}_f$-convergence = usual convergence
If $\mathcal{I}$ is admissible: $\mathcal{I}_f\text{-}\lim x_n = x \Rightarrow \mathcal{I}\text{-}\lim x_n = x$.

Definition 1.3. Let $\mathcal{I}$ be an ideal in $\mathbb{N}$ and X, Y be topological spaces. A map $f\colon X \to Y$ is called $\mathcal{I}$-*continuous* if for each sequence $(x_n)_{n=1}^{\infty}$ in X

$$\mathcal{I}\text{-}\lim x_n = x \quad \Rightarrow \quad \mathcal{I}\text{-}\lim f(x_n) = f(x)$$

holds.

2. Basic properties of $\mathcal{I}$-continuity

Theorem 2.2. *Let X, Y be topological spaces and let $\mathcal{I}$ be an arbitrary ideal in $\mathbb{N}$. If $f\colon X \to Y$ is continuous then f is $\mathcal{I}$-continuous.*

Theorem 2.3. *Let X, Y be topological spaces and let $\mathcal{I}$ be an arbitrary admissible ideal. If $f\colon X \to Y$ is $\mathcal{I}$-continuous then f is $\mathcal{I}_f$-continuous.*

continuity $\Rightarrow$ $\mathcal{I}$-continuity $\Rightarrow$ $\mathcal{I}_f$-continuity

A topological space is called *sequential* if a subset $V \subset X$ is closed in X whenever it contains with each convergent sequence all its limits. For sequential spaces $\mathcal{I}_f$-continuity and continuity are equivalent.

Corollary 2.4. *Let X be a sequential space and let $\mathcal{I}$ be an admissible ideal. Let Y be a topological space and let $f\colon X \to Y$ be a map. Then the following statements are equivalent:*

(1) f is continuous,

(2) f is $\mathcal{I}_f$-continuous,

(3) f is $\mathcal{I}$-continuous.

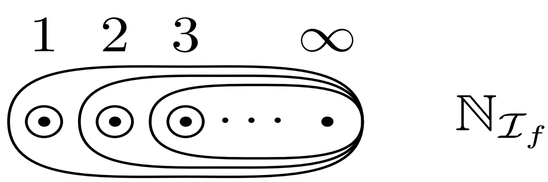
3. $\mathcal{I}$-continuity and prime spaces

A topological space X is a *prime space* if X has only one accumulation point. There is an one-to-one correspondence between prime spaces on the set $\mathbb{N} \cup \{\infty\}$ with the accumulation point ∞ and proper ideals in $\mathbb{N}$.

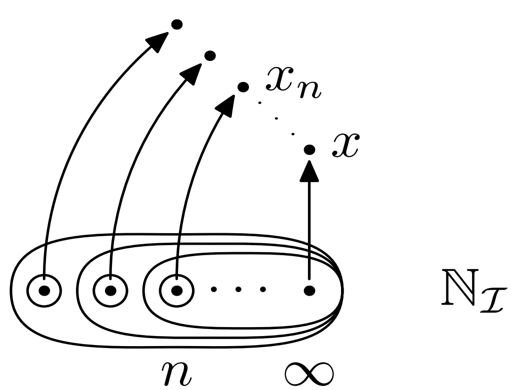
Let $\mathcal{I}$ be a proper ideal. We define a topological space $\mathbb{N}_{\mathcal{I}}$ on the set $\mathbb{N} \cup \{\infty\}$ as follows: $U \subset \mathbb{N} \cup \{\infty\}$ is open in $\mathbb{N}_{\mathcal{I}}$ if and only if $\infty \notin U$ or $U \setminus \{\infty\} \in \mathcal{F}(\mathcal{I})$.

If P is such a prime space then $\mathcal{I} = \{U \subset \mathbb{N} : U \text{ is closed in } P\}$ is a proper ideal.

Admissible ideals in $\mathbb{N}$ correspond to Hausdorff prime spaces.



Proposition 3.1. *Let X be a topological space, $x \in X$, $x_n \in X$ for each $n \in \mathbb{N}$. Let us define a map $f\colon \mathbb{N}_{\mathcal{I}} \to X$ by $f(n) = x_n$ and $f(\infty) = x$. Then $\mathcal{I}\text{-}\lim x_n = x$ if and only if f is continuous.*



Let $\mathcal{S}$ be a family of proper ideals in $\mathbb{N}$. We say that a topological space X is $\mathcal{S}$-*sequential* if every map $f\colon X \to Y$ is continuous provided that f is $\mathcal{I}$-continuous for each $\mathcal{I} \in \mathcal{S}$. (We briefly say that f is $\mathcal{S}$-*continuous*.)

Theorem 3.5. *A topological space X is $\mathcal{S}$-sequential if and only if it is the quotient of a topological sum of copies of spaces $\mathbb{N}_{\mathcal{I}}$, $\mathcal{I} \in \mathcal{S}$.*

A topological space is called *countable generated* if $V \subset X$ is closed whenever for each countable subspace U of X $V \cap U$ is closed in U.

Corollary 3.6. *Let $\mathcal{S}$ be the system of all (admissible) ideals in $\mathbb{N}$. Then a topological space X is countable generated if and only if X is $\mathcal{S}$-sequential, i.e. for every topological space Y and every map $f\colon X \to Y$ the following holds:*

f is continuous $\Leftrightarrow$ f is $\mathcal{I}$-continuous for each (admissible) ideal $\mathcal{I}$ in $\mathbb{N}$.

It is natural to ask whether $\mathcal{S}$-sequential spaces can be characterized similarly as the sequential spaces:

(2) V is closed in X if for each $\mathcal{I}$-convergent sequence $(x_n)_{n=1}^{\infty}$ of points of V, where $\mathcal{I} \in \mathcal{S}$, V contains all $\mathcal{I}$-limits of $(x_n)_{n=1}^{\infty}$.

Proposition 3.11. *Let $\mathcal{S}$ be a system of admissible ideals in $\mathbb{N}$. Let for each $\mathcal{I} \in \mathcal{S}$ the space $\mathbb{N}_{\mathcal{I}}$ fulfils (2). Then a topological space X is $\mathcal{S}$-sequential if and only if X fulfils (2).*

In general the condition (2) does not hold for the topological space $\mathbb{N}_{\mathcal{I}}$.

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