

# On $\mathcal{I}$ -convergence of series

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**Definition.** A series  $\sum_{n=1}^{\infty} a_n$  is said to be  $\mathcal{I}$ -convergent if the sequence  $(s_n)$  of its partial sums, i.e.,  $s_n = a_1 + a_2 + \cdots + a_n$ , is  $\mathcal{I}$ -convergent.

**Definition.** An admissible ideal  $\mathcal{I} \subseteq 2^{\mathbb{N}}$  has a property (T) if for every  $\mathcal{I}$ -convergent series  $\sum_{n=1}^{\infty} a_n$  there is a set  $F \in \mathcal{F}(\mathcal{I})$ ,  $F = \{n_1, n_2, \dots\}$ , such that the corresponding sub-series  $\sum_{k=1}^{\infty} a_{n_k}$  is convergent in the usual sense.

**Proposition (Červeňanský, Šalát).** Ideals  $\mathcal{I}_c, \mathcal{I}_d, \mathcal{I}_\delta, \mathcal{I}_u$  do not have property (T).

$$\mathcal{I}_c = \{A \subseteq \mathbb{N}; \sum_{k \in A} \frac{1}{k} < \infty\};$$

$$\mathcal{I}_d = \{A \subseteq \mathbb{N}; d(A) = 0\};$$

$$\mathcal{I}_\delta = \{A \subseteq \mathbb{N}; \delta(A) = 0\};$$

$$\mathcal{I}_u = \{A \subseteq \mathbb{N}; u(A) = 0\}.$$

## Sufficient conditions

**Definition.** An  $\mathcal{I}$ -partition of the set  $\mathbb{N}$  is a family of pairwise disjoint sets  $\mathcal{B} = \{B_n \in \mathcal{I}; n \in \mathbb{N}\}$  such that  $\bigcup_{n=1}^{\infty} B_n = \mathbb{N}$ .

**Theorem.** Let an admissible ideal  $\mathcal{I}$  satisfy the following conditions:

- (a) If  $A \in \mathcal{I}$  then  $A + 1 \in \mathcal{I}$ , where  $A + 1 = \{a + 1; a \in A\}$ .
- (b) For every  $\mathcal{I}$ -partition  $\mathcal{B}$  of  $\mathbb{N}$  there exists a set  $F \in \mathcal{F}(\mathcal{I})$  and  $M \in \mathbb{N}$  such that for all  $B \in \mathcal{B}$  we have  $|B \cap F| < M$ .

Then the ideal  $\mathcal{I}$  has the property (T).

**Proposition.** There is no admissible ideal satisfying both the conditions (a) and (b).

**Theorem.** *Let an admissible ideal  $\mathcal{I}$  satisfy the condition (a) and:*

(b') *There exists a sequence of positive reals  $(m_n)$  such that for every  $\mathcal{I}$ -partition  $\{B_n \in \mathcal{I}; n \in \mathbb{N}\}$  of  $\mathbb{N}$  there exists a set  $F \in \mathcal{F}(\mathcal{I})$  and  $M \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$  we have  $|B_n \cap F| < Mm_n$ .*

*Then the ideal  $\mathcal{I}$  has the property (T).*

**Proposition.** *The ideal  $\mathcal{I}_f$  does not satisfy the condition (b').*

## Necessary conditions

**Theorem.** *If an admissible ideal  $\mathcal{I}$  has the property (T) then it satisfies the condition (a).*

**Lemma.** *If  $\mathcal{I}$  is an admissible ideal satisfying the condition (a) then for every set  $A \in \mathcal{I}$  and for every  $M \in \mathbb{N}$  there is an  $a \in \mathbb{N}$  such that  $A \cap [a, a + M] = \emptyset$ . (A has arbitrary large gaps.)*

**Corollary.** *No maximal admissible ideal satisfies the condition (a) – and hence no maximal ideal has the property (T).*

**Theorem.** *If an admissible ideal  $\mathcal{I}$  has the property (T) then it satisfies the condition*

(c) *For every  $\mathcal{I}$ -partition  $\mathcal{B}$  of  $\mathbb{N}$  there exists a set  $F \in \mathcal{F}(\mathcal{I})$  such that the set  $B \cap F$  is finite for all  $B \in \mathcal{B}$ .*

**Definition.** An admissible ideal  $\mathcal{I} \subseteq 2^{\mathbb{N}}$  is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets  $\{B_1, B_2, \dots\}$  belonging to  $\mathcal{I}$  there exists a countable family of sets  $\{A_1, A_2, \dots\}$  such that the set  $A_n \Delta B_n$  is finite for all  $n \in \mathbb{N}$  and  $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{I}$ .

**Proposition (Kostyrko, Šalát, Wilczyński).**

*An admissible ideal  $\mathcal{I}$  satisfies the (AP) condition iff for every  $\mathcal{I}$ -convergent sequence  $(x_n)_{n \in \mathbb{N}}$  there exists a set  $F \in \mathcal{F}(\mathcal{I})$ ,  $F = \{n_1 < n_2 < \dots\}$  such that the corresponding subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  is convergent in the usual sense and  $\mathcal{I}\text{-}\lim_{n \rightarrow \infty} x_n = \lim_{k \rightarrow \infty} x_{n_k}$ .*

**Proposition.** *An admissible ideal  $\mathcal{I}$  satisfies the (AP) condition iff  $\mathcal{I}$  satisfies the condition (c).*

**Proposition.** *Every admissible ideal  $\mathcal{I}$  containing an infinite set and satisfying the conditions (a) and (c) contains an admissible subideal  $\mathcal{J} \subseteq \mathcal{I}$  which does not satisfy the condition (c).*