On $\mathcal{I}$-convergence of series

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Workshop on the Density Concept
May 16 – 18, 2004
Bratislava

Definition. A series $\sum_{n=1}^{\infty} a_n$ is said to be $\mathcal{I}$-convergent if the sequence (s_n) of its partial sums, i.e., $s_n = a_1 + a_2 + \cdots + a_n$, is $\mathcal{I}$-convergent.

Definition. An admissible ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ *has a property* (T) if for every $\mathcal{I}$-convergent series $\sum_{n=1}^{\infty} a_n$ there is a set $F \in \mathcal{F}(\mathcal{I})$, $F = \{n_1, n_2, \dots\}$, such that the corresponding subseries $\sum_{k=1}^{\infty} a_{n_k}$ is convergent in the usual sense.

Proposition (Červeňanský, Šalát). *Ideals $\mathcal{I}_c, \mathcal{I}_d, \mathcal{I}_\delta, \mathcal{I}_u$ do not have property* (T)*.*

$$\mathcal{I}_c = \{A \subseteq \mathbb{N};\ \sum_{k \in A} \frac{1}{k} < \infty\};$$
$$\mathcal{I}_d = \{A \subseteq \mathbb{N};\ d(A) = 0\};$$
$$\mathcal{I}_\delta = \{A \subseteq \mathbb{N};\ \delta(A) = 0\};$$
$$\mathcal{I}_u = \{A \subseteq \mathbb{N};\ u(A) = 0\}.$$

Sufficient conditions

Definition. An *$\mathcal{I}$-partition of the set* $\mathbb{N}$ is a family of pairwise disjoint sets $\mathcal{B} = \{B_n \in \mathcal{I};\ n \in \mathbb{N}\}$ such that $\bigcup\limits_{n=1}^{\infty} B_n = \mathbb{N}$.

Theorem. *Let an admissible ideal $\mathcal{I}$ satisfy the following conditions:*

(a) *If $A \in \mathcal{I}$ then $A + 1 \in \mathcal{I}$, where $A + 1 = \{a + 1;\ a \in A\}$.*

(b) *For every $\mathcal{I}$-partition $\mathcal{B}$ of $\mathbb{N}$ there exists a set $F \in \mathcal{F}(\mathcal{I})$ and $M \in \mathbb{N}$ such that for all $B \in \mathcal{B}$ we have $|B \cap F| < M$.*

Then the ideal $\mathcal{I}$ has the property (T)*.*

Proposition. *There is no admissible ideal satisfying both the conditions* (a) *and* (b)*.*

Theorem. *Let an admissible ideal $\mathcal{I}$ satisfy the condition* (a) *and:*

(b') *There exists a sequence of positive reals (m_n) such that for every $\mathcal{I}$-partition $\{B_n \in \mathcal{I}; n \in \mathbb{N}\}$ of $\mathbb{N}$ there exists a set $F \in \mathcal{F}(\mathcal{I})$ and $M \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have $|B_n \cap F| < M m_n$.*

Then the ideal $\mathcal{I}$ has the property (T)*.*

Proposition. *The ideal $\mathcal{I}_f$ does not satisfy the condition* (b')*.*

Necessary conditions

Theorem. *If an admissible ideal $\mathcal{I}$ has the property* (T) *then it satisfies the condition* (a)*.*

Lemma. *If $\mathcal{I}$ is an admissible ideal satisfying the condition* (a) *then for every set $A \in \mathcal{I}$ and for every $M \in \mathbb{N}$ there is an $a \in \mathbb{N}$ such that $A \cap [a, a + M] = \emptyset$. ($A$ has arbitrary large gaps.)*

Corollary. *No maximal admissible ideal satisfies the condition* (a) *− and hence no maximal ideal has the property* (T)*.*

Theorem. *If an admissible ideal* $\mathcal{I}$ *has the property* (T) *then it satisfies the condition*

(c) *For every* $\mathcal{I}$*-partition* $\mathcal{B}$ *of* $\mathbb{N}$ *there exists a set* $F \in \mathcal{F}(\mathcal{I})$ *such that the set* $B \cap F$ *is finite for all* $B \in \mathcal{B}$.

Definition. An admissible ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to satisfy the condition (*AP*) if for every countable family of mutually disjoint sets $\{B_1, B_2, \dots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{A_1, A_2, \dots\}$ such that the set $A_n \triangle B_n$ is finite for all $n \in \mathbb{N}$ and $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{I}$.

Proposition (Kostyrko, Šalát, Wilczyński). *An admissible ideal* $\mathcal{I}$ *satisfies the* (AP) *condition iff for every* $\mathcal{I}$*-convergent sequence* $(x_n)_{n\in\mathbb{N}}$ *there exists a set* $F \in \mathcal{F}(\mathcal{I})$, $F = \{n_1 < n_2 < \ldots\}$ *such that the corresponding subsequence* $(x_{n_k})_{k\in\mathbb{N}}$ *is convergent in the usual sense and* $\mathcal{I}\text{-}\lim\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} x_{n_k}$.

Proposition. *An admissible ideal* $\mathcal{I}$ *satisfies the* (AP) *condition iff* $\mathcal{I}$ *satisfies the condition* (c).

Proposition. *Every admissible ideal* $\mathcal{I}$ *containing an infinite set and satisfying the conditions* (a) *and* (c) *contains an admissible subideal* $\mathcal{J} \subseteq \mathcal{I}$ *which does not satisfy the condition* (c).