A NOTE ON THE CHARACTERISTIC RANK AND RELATED NUMBERS

EUĐOVÍT BALKO* AND JÚLIUS KORBAŠ**

Dedicated to Professor Masaharu Morimoto on the occasion of his 60th birthday

Abstract. This note quantifies, via a sharp inequality, an interplay between (a) the characteristic rank of a vector bundle over a topological space $X$, (b) the $\mathbb{Z}_2$-Betti numbers of $X$, and (c) sums of the numbers of certain partitions of integers. In a particular context, (c) is transformed into a sum of the readily calculable Betti numbers of the real Grassmann manifolds.

1. Introduction

The characteristic rank of a smooth closed connected $d$-dimensional manifold $M$ was introduced, and in some cases also calculated, in [3] as the largest integer $k$, $0 \leq k \leq d$, such that each cohomology class in $H^j(M;\mathbb{Z}_2)$ with $j \leq k$ can be expressed as a polynomial in the Stiefel-Whitney classes of $M$; further results can be found in [2]. Recently, A. Naolekar and A. Thakur ([6]) have adapted this homotopy invariant of smooth closed connected manifolds to vector bundles.

Given a path-connected topological space $X$ and a real vector bundle $\alpha$ over $X$, by the characteristic rank of $\alpha$, denoted charrank$(\alpha)$, we understand the largest $k$, $0 \leq k \leq \dim_{\mathbb{Z}_2}(X)$, such that every cohomology class in $H^j(X;\mathbb{Z}_2)$, $0 \leq j \leq k$, can be expressed as a polynomial in the Stiefel-Whitney classes $w_i(\alpha) \in H^{i}(X;\mathbb{Z}_2)$; $\dim_{\mathbb{Z}_2}(X)$ is the supremum of all $q$ such that the cohomology groups $H^q(X;\mathbb{Z}_2)$ do not vanish. We shall always use $\mathbb{Z}_2$ as the coefficient group for cohomology, and so we shall write $H^i(X)$ instead of $H^i(X;\mathbb{Z}_2)$ and $\dim(X)$ instead of $\dim_{\mathbb{Z}_2}(X)$. Of course, if $TM$ denotes the tangent bundle of $M$, then we have charrank$(TM) = \text{charrank}(M)$. Results on the characteristic rank of vector bundles over the Stiefel manifolds can be found in [4]. In addition to being an interesting question in its own right, there are other reasons for investigating the characteristic rank; one of them is its close relation to the cup-length of a given space ([3], [6]).

This note presents a result on the characteristic rank that may also prove useful in some non-topological questions (for instance, in the theory of partitions; see [1]). More precisely, we quantify (Theorem 2.1), via a sharp inequality, an interplay between (a) the characteristic rank of a vector bundle, (b) the $\mathbb{Z}_2$-Betti numbers of base space of this vector bundle, and (c) sums of the numbers of certain partitions of integers. In a particular context, (c) is transformed into a sum of the readily calculable Betti numbers of the real Grassmann manifolds.

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calculable Betti numbers of the real Grassmann manifolds $G_{n,k}$ of all $k$-dimensional vector subspaces in $\mathbb{R}^n$.

2. The result and its proof

Let $p(a, b, c)$ be the number of partitions of $c$ into at most $b$ parts, each $\leq a$. At the same time, given a subset $A$ of the positive integers, let $p(A, b, c)$ denote the number of partitions of $c$ into $b$ parts each taken from the set $A$. We recall that $(\{5, 6, 7\}) p(a, b, c)$ is the same as the number of cells of dimension $c$ in the Schubert cell decomposition of the Grassmann manifold $G_{a+b, b}$, or the same as the $\mathbb{Z}_2$-Betti number $b_c(G_{a+b, b}) = b_c(G_{a+b, a})$. Of course, for dimensional reasons, $b_c(G_{a+b, b}) = p(a, b, c) = 0$ for $c > ab = \dim(G_{a+b, b})$. In addition, we denote by $p(c)$ the total number of partitions of $c$. Even if not explicitly stated, $X$ will always mean a path-connected topological space.

For obvious reasons, we confine our considerations to those vector bundles having total Stiefel-Whitney class non-trivial. For a given space $X$, an ordered subset in \{1, 2, \ldots, \dim(X)\}, with the least element (denoted by) $\nu$ and the greatest element (denoted by) $\kappa$, will be denoted by $S_{\nu}^\kappa$. If, in addition, $\alpha$ is a real vector bundle
over $X$, then $S_{\nu}^\kappa(\alpha)$ will denote any $S_{\nu}^\kappa \neq \emptyset$ such that $w_i(\alpha) = 0$ for every positive $i \notin S_{\nu}^\kappa$. In general, there are several possible choices of $S_{\nu}^\kappa(\alpha)$: one can always take $S_{\nu}^\kappa(\alpha) = \{1 = \nu, 2, \ldots, \dim(X) = \kappa\}$, but if we know, for instance, that $w_3(\alpha) = 0$, then we can also take the set $\{1 = \nu, 2, 4, \ldots, \dim(X) = \kappa\}$ in the role of $S_{\nu}^\kappa(\alpha)$. Now we can state and prove our result.

**Theorem 2.1.** Let $\alpha$ be a real vector bundle over a path-connected topological space $X$ such that $w(\alpha) \neq 1$, and let $\operatorname{charrank}(\alpha) \geq t$ for some $t$. Then, for every $j \in \{1, \ldots, t\}$, we have an inequality,

$$b_j(X) \leq \sum_{s=1}^{\lfloor \frac{\mu}{\nu} \rfloor} p(\{x \in S_{\nu}^\kappa(\alpha)|x \leq \mu\}, s, j),$$

(1)

where $\mu = \min\{j, \kappa\}$.

In particular, if the set $\{x \in S_{\nu}^\kappa(\alpha)|x \leq \mu\}$ in (1) is gapless, then (1) turns into

$$b_j(X) \leq \sum_{\frac{\mu}{\nu} \leq s \leq \frac{\mu}{\nu}} b_{j-s}(G_{\mu-\nu+s,s}).$$

(2)

**Proof.** Let us fix one of the sets $S_{\nu}^\kappa(\alpha)$. If $\operatorname{charrank}(\alpha) \geq t$ for some $t$ then, for every $j \in \{1, 2, \ldots, t\}$, the $\mathbb{Z}_2$-vector space $H^j(X)$ is spanned by all the products of the form

$$w_{\nu}^{i_1}(\alpha) \cup \cdots \cup w_{\nu}^{i_j}(\alpha) \in H^j(X).$$

(3)

Since $\nu(i_\nu + \cdots + i_\mu) \leq \nu i_\nu + \cdots + \nu i_\mu = j$ (thus $i_\nu + \cdots + i_\mu \leq \frac{j}{\nu}$), the number of generators of the form (3), that is, an upper bound for $b_j(X)$, is the number of partitions of $j$ into at most $\lfloor \frac{j}{\nu} \rfloor$ positive parts each taken from the set $\{x \in S_{\nu}^\kappa(\alpha)|x \leq \mu\}$. In other words, this upper bound is

$$\sum_{s=1}^{\lfloor \frac{\mu}{\nu} \rfloor} p(\{x \in S_{\nu}^\kappa(\alpha)|x \leq \mu\}, s, j)$$

as was asserted.
In order to transform (1) into (2) when the set \( \{ x \in S^\nu_0 | x \leq \mu \} \) in (1) is
gapless, it suffices to know that
\[
\sum_{s=1}^{\lfloor \frac{j}{2} \rfloor} p(\nu, \nu + 1, \nu + 2, \ldots, \mu, s, j) = \sum_{s=1}^{\lfloor \frac{j}{2} \rfloor} p(\mu - \nu, s, j - \nu s);
\] (4)

this equality is verified by the following elementary considerations.

Let \( P(j)_{\nu, \nu+1, \ldots, \mu} \) be the set of partitions of \( j \) into \( x \) positive parts each taken
from the set \( \{ \nu, \nu + 1, \ldots, \mu \} \) and let \( P(l)_{1, 2, \ldots, \mu - \nu} \) be the set of partitions of \( l \) (\( l \geq 0 \)) into at most \( x \) parts each taken from the set \( \{ 1, 2, \ldots, \mu - \nu \} \) (the set \( P(0)_{1, 2, \ldots, \mu - \nu} \) for all \( x > 0 \) and \( \mu - \nu > 0 \) contains just one element, namely
the empty partition). Of course, the total number of elements in \( P(l)_{1, 2, \ldots, \mu - \nu} \) is
\( p(\mu - \nu, x, l) \).

Each element of \( P(j)_{\nu, \nu+1, \ldots, \mu} \) has the form
\[ a_1 + \cdots + a_{i(\nu)} + a_{i(\nu)+1} + \cdots + a_x, \]
where \( i(\nu) \geq 0, a_1 = \cdots = a_{i(\nu)} = \nu < a_{i(\nu)+1} \leq a_{i(\nu)+2} \leq \cdots \leq a_x \)
and \( a_1 + \cdots + a_{i(\nu)} + a_{i(\nu)+1} + \cdots + a_x = j \). The map
\[ P(j)_{\nu, \nu+1, \ldots, \mu} \rightarrow P(j - \nu x)_{1, 2, \ldots, \mu - \nu}, \]
\[ a_1 + \cdots + a_{i(\nu)} + a_{i(\nu)+1} + \cdots + a_x \mapsto (a_{i(\nu)+1} - \nu) + (a_{i(\nu)+2} - \nu) + \cdots + (a_x - \nu) \]
is bijective; indeed, the inverse map is readily seen to be
\[ P(j - \nu x)_{1, 2, \ldots, \mu - \nu} \rightarrow P(j)_{\nu, \nu+1, \ldots, \mu}, \]
\[ b_1 + b_2 + \cdots + b_{x-1} \mapsto \nu + \cdots + \nu + (b_1 + \nu) + \cdots + (b_{x-1} + \nu). \]

Thus (4) is verified, and the proof of Theorem 2.1 is finished.

\[ \square \]

Example 2.2. We recall ([5, Theorem 7.1]) that the cohomology algebra of the
infinite Grassmannian \( G_{\infty, k} \) can be identified with a polynomial algebra,
\[ H^*(G_{\infty, k}) = \mathbb{Z}_2[w_1, \ldots, w_k], \]
where \( w_i \in H^i(G_{\infty, k}) \) is the \( i \)th Stiefel-Whitney class of the universal \( k \)-plane
bundle \( \gamma_{\infty, k} \). Thus \( \text{charrank}(G_{\infty, k}) = \infty \) and we may take \( S^\nu_0(\gamma_{\infty, k}) = \{ 1 = \nu, 2, \ldots, k \} \).
Since for \( X = G_{\infty, k} \) there are no relations among the generators of the form (3), inequalities (1) and (2) turn into one of the following equalities for
any positive integer \( j \):
\[ b_j(G_{\infty, k}) = p(j) = \sum_{s=1}^{\lfloor j/2 \rfloor} b_{j-s}(G_{j-1+s, s}) \text{ if } j \leq k, \] (5)
\[ b_j(G_{\infty, k}) = \sum_{s=1}^{j} p(1, 2, \ldots, k, s, j) = \sum_{s=1}^{\lfloor j/2 \rfloor} b_{j-s}(G_{k-1+s, s}) \text{ if } j > k. \] (6)

In a similar way, one can see that (1) and (2) are also sharp for \( X = \tilde{G}_{\infty, k} \), the
infinite oriented Grassmannian.
References


*Faculty of Mechanical Engineering of SUT, Námestie slobody 17, SK-812 31 Bratislava, Slovakia
E-mail address: ludovit.balko@gmail.com*

**Faculty of Mathematics, Physics, and Informatics, Comenius University, Mlynská dolina, SK-842 48 Bratislava, Slovakia or Mathematical Institute of SAS, Štefánikova 49, SK-841 73 Bratislava, Slovakia
E-mail address: korbas@fmph.uniba.sk*