An explicit formula for the cup-length of the rotation group

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Abstract

This paper gives an explicit formula for the $\mathbb{Z}_2$-cup-length of the rotation group $\text{SO}(n)$.

1 Introduction and the main result

As is well known, the $\mathbb{Z}_2$-cup-length $\text{cup}(X; \mathbb{Z}_2)$ of a compact path-connected topological space $X$ is the maximum of all integers $c$ such that there exist reduced cohomology classes $a_1, \ldots, a_c \in \tilde{H}^*(X; \mathbb{Z}_2)$ such that their cup product $a_1 \cup \cdots \cup a_c$ does not vanish. Instead of the usual notation $a \cup b$, we shall write $ab$, $H^*(X; \mathbb{Z}_2)$ will be abbreviated to $H^*(X)$, and $\text{cup}(X; \mathbb{Z}_2)$ will be shortened to $\text{cup}(X)$ in the sequel (we shall only consider cohomology with coefficients in $\mathbb{Z}_2$). The Elsholz inequality $\text{cat}(X) \geq \text{cup}(X)$ relates $\text{cup}(X)$ to another important homotopy invariant, the Lyusternik-Shnirel’man category $\text{cat}(X)$; the latter is defined to be the least positive integer $k$ such that $X$ can be covered by $k + 1$ open subsets each of which is contractible in $X$.

For the rotation (or special orthogonal) group $\text{SO}(n)$, the $\mathbb{Z}_2$-cohomology algebra is known due to A. Borel [1]. We recall its description by A. Hatcher [2]:

$$H^*(\text{SO}(n)) \cong \bigotimes_{i \text{ odd}, i < n} \mathbb{Z}_2[\beta_i] / (\beta_i^{p_i})$$

(1)

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where the degree of $\beta_i$ is equal to $i$ and $p_i$ is the smallest power of 2 such that the degree of $\beta_{i}^{p_i}$ is at least $n$. This cohomology algebra looks quite simple but, to the best of the author’s knowledge, only recursive formulas for $\text{cup}(\text{SO}(n))$ were known up to now: the formula $\text{cup}(\text{SO}(2n)) = 2\text{cup}(\text{SO}(n)) + n$, $\text{cup}(\text{SO}(2n)) = \text{cup}(\text{SO}(2n-1)) + 1$, known to the author thanks to Mamoru Mimura, from a 2008-preprint by Kei Sugata, and the formula (perhaps folkloric) $\text{cup}(\text{SO}(n+1)) = \text{cup}(\text{SO}(n)) + 2^{v_2(n)}$, where $2^{v_2(n)}$ is the highest power of 2 dividing $n$.

But the problem of finding an explicit formula for $\text{cup}(\text{SO}(n))$ was open thus far. The main aim of this note is to solve it by proving that the cup-length of $\text{SO}(n)$ can be expressed in the following surprisingly concise way.

**Theorem 1.1.** For any positive integer $n$,

$$\text{cup}(\text{SO}(n)) = n - 1 + (n - 1)'$$

where $(n - 1)' = \sum_{i=1}^{k} i n_i 2^{i-1}$ if $n - 1$ has the dyadic expansion $\sum_{i=0}^{k} n_i 2^i$.

In view of the Elsholz inequality, Theorem 1.1 immediately implies a global lower bound for the Lyusternik-Shnirel’man category of rotation groups.

**Corollary 1.1.** We have

$$\text{cat}(\text{SO}(n)) \geq n - 1 + (n - 1)'$$

Due to I. James and W. Singhof [5], N. Iwase, M. Mimura, and T. Nishimoto [3], and N. Iwase, K. Kikuchi, and T. Miyauchi [4], it is known that this lower bound is sharp for $n = 1, 2, \ldots, 10$. Of course, our formula for $\text{cup}(\text{SO}(n))$ (Theorem 1.1) is of interest in its own right. But it also enables us to transform the conjecture worded in [4], “this would suggest that $\text{cat}(\text{SO}(n)) = \text{cup}(\text{SO}(n))$ for all $n$,” into the following explicit problem.

**Question 1.1.** Is it true that $\text{cat}(\text{SO}(n)) = n - 1 + (n - 1)'$ for $n \geq 1$?

For odd $n$ ($n \geq 3$), let $q$ be the unique integer such that $2^{q-1} < n < 2^q$. Write $n = 1 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_t}$ ($1 \leq v_1 < v_2 < \cdots < v_t$) the dyadic expansion of $n$. Then we have $v_t < q$ and Theorem 1.1 yields that $\text{cup}(\text{SO}(n)) < \frac{(n-1)(q+2)}{2}$. In a similar way, one verifies that $\text{cup}(\text{SO}(n)) \leq \frac{(n-2)(q+2)}{2}$ for even $n$. [It is easy to compare these bounds with $\frac{n(n-1)}{2} = \left(\begin{array}{c}n \\ 2 \end{array}\right) = \dim(\text{SO}(n))$.] We thus may state the following weaker (but presumably still very hard) question (whose answer by “No” would of course mean that also Question 1.1 must be answered by “No”).

**Question 1.2.** For a positive integer $n$, let $q$ denote the unique integer such that $2^{q-1} < n \leq 2^q$. Is it true that $\text{cat}(\text{SO}(n)) < \frac{(n-1)(q+2)}{2}$ for all odd $n$, $n \geq 11$, and $\text{cat}(\text{SO}(n)) \leq \frac{(n-2)(q+2)}{2}$ for all even $n$, $n \geq 12$?
2 Proof of the main result

Poincaré duality implies that the cup-length of SO(n) is realized by a cohomology class in the top degree; note that we may identify \( H^{\frac{n(n-1)}{2}}(\text{SO}(n)) = \mathbb{Z}_2 \). Obviously, if \( n \) is odd, then \( \text{cup}(\text{SO}(n)) \) equals the sum of the exponents in

\[
\beta_1^{p_1 - 1} \beta_3^{p_3 - 1} \cdots \beta_{n-4}^{p_{n-4} - 1} \beta_{n-2} \in H^{\frac{n(n-1)}{2}}(\text{SO}(n)).
\]

Thus by (2), for odd \( n \), we see that \( \text{cup}(\text{SO}(n)) = (p_1 - 1) + (p_3 - 1) + \cdots + (p_{n-4} - 1) + 1 \); consequently, \( \text{cup}(\text{SO}(n+1)) \) is obviously the sum of the exponents in the product \( \beta_1^{p_1 - 1} \beta_3^{p_3 - 1} \cdots \beta_{n-4}^{p_{n-4} - 1} \beta_{n-2} \beta_n \), since \( \text{dim}(\text{SO}(n+1)) - \text{dim}(\text{SO}(n)) = n \) (this difference equals the degree in which we have the generator \( \beta_n \)). Thus indeed, for odd \( n \), \( \text{cup}(\text{SO}(n+1)) = [(p_1 - 1) + (p_3 - 1) + \cdots + (p_{n-4} - 1) + 1] + 1 = \text{cup}(\text{SO}(n)) + 1 \), as claimed. For even \( n \), a proof is omitted. We have come to the following fact.

**Fact 2.1.** Let \( c(n) = \text{cup}(\text{SO}(n)) \), \( n \geq 1 \), and \( v_2(n) \) be the exponent of the highest power of 2 dividing \( n \). Then (i) \( c(1) = 0 \); (ii) \( c(n+1) = c(n) + 2^{v_2(n)} \).

The key observation is the following.

**Lemma 2.1.** We have \( c(m + 2^k) - c(m) = c(2^k - 1) + 2^k + 1 \), if \( 1 \leq m \leq 2^k, k \geq 1 \).

**Proof.** If \( m = 1 \), we have \( c(1 + 2^k) - c(1) = c(1 + 2^k) = c(2^k) + 2^k = c(2^k - 1) + 2^k + 1 \) by Fact 2.1 (i) and (ii), and so we assume \( 1 < m \leq 2^k \). By Fact 2.1 (ii), we have \( c(m + 2^k) - c(m - 1 + 2^k) = 2^{v_2(m-1)} = c(m) - c(m - 1) \), and thus obtain \( c(m + 2^k) - c(m) = c(m - 1 + 2^k) - c(m - 1) = \cdots = c(1 + 2^k) - c(1) \) and is equal to \( c(2^k - 1) + 2^k + 1 \).

To show the main result, we need the following proposition.

**Proposition 2.1.** For any \( k \geq 1 \), \( c(2^k - 1) = k2^{k-1} - 1 \).

**Proof.** By Lemma 2.1 with \( m = 2^k - 1 \), we obtain \( 1 \leq m \leq 2^k \) and \( c(2^k + 1 - 1) = c(2^k - 1 + 2^k) = c(2^k - 1) + c(2^k - 1) + 2^k + 1 \), which yields the following recurrence relation by taking \( a_k = \frac{c(2^k - 1) + 1}{2^k} \):

\[
a_{k+1} = a_k + \frac{1}{2}, \quad k \geq 1,
\]

which is an arithmetic sequence starting with \( a_1 = \frac{c(2^1 - 1) + 1}{2^1} = \frac{1}{2} \), and hence \( a_k = \frac{k}{2} \) and \( c(2^k - 1) = k2^{k-1} - 1, \quad k \geq 1 \).

Under the above observation, we obtain the main result as follows.
Theorem. We have $c(n) = n - 1 + (n - 1)'$, $n \geq 1$, where $(n - 1)' = \sum_{i=1}^{k} i n_i 2^{i-1}$ if $n - 1$ has the dyadic expansion $\sum_{i=0}^{k} n_i 2^i$.

Proof. If $n = 1$, it is clear by Fact 2.1 (i), and so we assume $n \geq 2$ and $n - 1 \geq 1$ has the dyadic expansion $\sum_{i=0}^{k} n_i 2^i$, with $n_k = 1$. We show the formula by induction on $k \geq 0$.

$k = 0$: Then $n = 2$ and $c(2) = 1 = 1 + 1'$ by Fact 2.1 (i) and (ii).

$k \geq 1$: Let $m = n - 2^k$, to obtain $0 \leq m < 2^k$ and $c(m) = m - 1 + (m - 1)'$ by induction hypothesis. Then by Lemma 2.1, we have $c(n) = c(m + 2^k) = c(m) + c(2^k - 1) + 2^k + 1 = m - 1 + (m - 1)' + k 2^{k-1} + 2^k = (n - 1) + (n - 1)'$. This completes the proof of Theorem.  

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References


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