$\mathcal{I}^{\mathcal{K}}$-CONVERGENCE

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Abstract. In this paper we introduce $\mathcal{I}^{\mathcal{K}}$-convergence which is a common generalization of the $\mathcal{I}^*$-convergence of sequences, double sequences and nets. We show that many results that were shown before for these special cases are true for the $\mathcal{I}^{\mathcal{K}}$-convergence, too.

1. Historical background and introduction

The main topic of this paper is convergence of a function along an ideal. As the dual notion of the convergence along a filter was studied as well, let us start by saying a few words about the history of this concept.

It was defined for the first time probably by Henri Cartan [6] (see also [5, p.71, Definition 1]). Although the notion of a limit along a filter was defined here in the maximal possible generality – the considered filter could be a filter on an arbitrary set and the limit was defined for any map from this set to a topological space – the attention of mathematicians in the following years was mostly focused to two special cases.

In general topology the notion of the limit of a filter on a topological space X became one of the two basic tools used to describe the convergence in general topological spaces together with the notion of a net (see [12, Section 1.6]).

Some authors studied also the convergence of a sequence along a filter. This notion was rediscovered independently by several authors, we could mention A. Robinson [34], A. R. Bernstein [4] (these authors used ultrafilters only) or M. Katětov [21].

The definition of the limit along a filter can be reformulated using ideals – the dual notion to the notion of filter. This type of limit of sequences was introduced independently by P. Kostyrko, M. Mačaj and T. Šalát [22] and F. Nuray and W. H. Ruckle [32] and studied under the name *$\mathcal{I}$-convergence* of a sequence by several authors (see also [10, 23, 24]). The motivation for this direction of research was an effort to generalize some known results on statistical convergence. Since the notions that we intend to generalize in this paper stem from one of the results on the statistical convergence, let us describe in more detail how they evolved.

Motivated by a result of T. Šalát [35] and J. A. Fridy [15] about statistically convergent sequences, the authors of [22] also defined so called $\mathcal{I}^*$-convergence (a sequence $(x_n)_{n=1}^\infty$ being *$\mathcal{I}^*$-convergent* to x provided that there exist $M \in \mathcal{F}(\mathcal{I})$ such that the corresponding subsequence converges to x) and asked for which ideals the notions of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence coincide. This question was answered in [24] where the authors showed that these notions coincide if and only if the ideal $\mathcal{I}$ satisfies the property AP, which we call $\mathrm{AP}(\mathcal{I}, \mathrm{Fin})$ here (see also [23, 32]).

Later the analogues of the notion of $\mathcal{I}^*$-convergence were defined and similar characterizations were obtained for double sequences (see [8, 25]) and nets (see [29]).

In this paper we define $\mathcal{I}^{\mathcal{K}}$-convergence as a common generalization of all these types of $\mathcal{I}^*$-convergence and obtain results which strengthen the results from the above papers. In the last we also point at neglected relation between the $\mathcal{I}$-convergence of sequences and double sequences.

Although our motivation arises mainly from the results obtained for sequences, we will work with functions. One of the reasons is that using functions sometimes helps to simplify notation.

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Another reason is that we tried to obtain the maximal possible generality allowed by the tools we are using.

2. Notation and preliminaries

In this section we recall some notions and results concerning the $\mathcal{I}$-convergence.

If S is a set, then a system $\mathcal{I} \subseteq \mathcal{P}(S)$ is called an *ideal on* S if it is additive, hereditary and non-empty, that is,

(i) $\emptyset \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I} \wedge B \subseteq A \Rightarrow B \in \mathcal{I}$.

An ideal on S is called *admissible* if it contains all singletons, that is, $\{s\} \in \mathcal{I}$ for each $s \in S$. An ideal $\mathcal{I}$ on S is called *proper* if $S \notin \mathcal{I}$, a proper ideal is called *maximal* if it is a maximal element of the set of all proper ideals on S ordered by inclusion. It can be shown that a proper ideal $\mathcal{I}$ is maximal if and only if $(\forall A \subseteq S)\ A \in \mathcal{I} \vee S \setminus A \in \mathcal{I}$.

We will denote by Fin the ideal of all finite subsets of a given set S.

The dual notion to the notion of an ideal is the notion of a filter. A system $\mathcal{F} \subseteq \mathcal{P}(S)$ of subsets of S is called a *filter on* S if

(i) $S \in \mathcal{F}$,
(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F} \wedge B \supseteq A \Rightarrow B \in \mathcal{F}$.

A filter $\mathcal{F}$ is called *proper* if $\emptyset \notin \mathcal{F}$.

The dual notion to the notion of a maximal ideal is the notion of *ultrafilter.*

A system $\mathcal{B} \subseteq \mathcal{P}(S)$ is called *filterbase* if

(i) $\mathcal{B} \neq \emptyset$,
(ii) $A, B \in \mathcal{B} \Rightarrow (\exists C \in \mathcal{B})\ C \subseteq A \cap B$.

If $\mathcal{B}$ is a filterbase, then the system

$$\mathcal{F} = \{A \supseteq B; B \in \mathcal{B}\}$$

is a filter. It is called filter *generated* by the base $\mathcal{B}$.

For any ideal $\mathcal{I}$ on a set S the system

$$\mathcal{F}(\mathcal{I}) = \{X \setminus A; A \in \mathcal{I}\}$$

is a filter on S. It is called the *filter associated with the ideal* $\mathcal{I}$. In a similar way we can obtain ideal from any filter. This yields a one-to-one correspondence between ideals and filters on a given set.

Definition 2.1. Let $\mathcal{I}$ be an ideal on a set S and X be a topological space. A function $f \colon S \to X$ is said to be *$\mathcal{I}$-convergent to $x \in X$* if

$$f^{-1}(U) = \{s \in S; f(s) \in U\} \in \mathcal{F}(\mathcal{I})$$

holds for every neighborhood U of the point x.

We use the notation

$$\mathcal{I}\text{-}\lim f = x.$$

If $S = \mathbb{N}$ we obtain the usual definition of $\mathcal{I}$-convergence of sequences. In this case the notation $\mathcal{I}\text{-}\lim x_n = x$ is used.

We include a few basic facts concerning $\mathcal{I}$-convergence for future reference.

Lemma 2.2. *Let S be a set, let $\mathcal{I}$, $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals on S and let X and Y be topological spaces.*

(i) *If $\mathcal{I}$ is not proper, that is, if $\mathcal{I} = \mathcal{P}(S)$, then every function $f \colon S \to X$ converges to each point of X.*
(ii) *If $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then for every function $f \colon S \to X$, we have*

$$\mathcal{I}_1\text{-}\lim f = x \qquad \textit{implies} \qquad \mathcal{I}_2\text{-}\lim f = x.$$

(iii) *If X is Hausdorff and $\mathcal{I}$ is proper, then every function $f\colon S \to X$ has at most one $\mathcal{I}$-limit.*
(iv) *If $g\colon X \to Y$ is a continuous mapping and $f\colon S \to X$ is $\mathcal{I}$-convergent to x, then $g \circ f$ is $\mathcal{I}$-convergent to $g(x)$.*
(v) *If $\mathcal{I}$ is a maximal ideal and X is compact, then every function $f\colon S \to X$ has an $\mathcal{I}$-limit.*

Let us note that the above properties are more frequently stated for filters rather than ideals. Moreover, the property (iii) is in fact a characterization of Hausdorff spaces and the property (v) is a characterization of compact spaces.

3. $\mathcal{I}^{\mathcal{K}}$-CONVERGENCE

3.1. **Definition and basic results.** As we have already mentioned, we aim to generalize the notion of $\mathcal{I}^*$-convergence of sequences, introduced in [22] for sequences of real numbers and generalized to metric spaces in [24]. Since we are working with functions, we modify this definition in the following way:

Definition 3.1. Let $\mathcal{I}$ be an ideal on a set S and let $f\colon S \to X$ be a function to a topological space X. The function f is called *$\mathcal{I}^*$-convergent* to the point x of X if there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that the function $g\colon S \to X$ defined by

$$g(s) = \begin{cases} f(s), & \text{if } s \in M \\ x, & \text{if } s \notin M \end{cases}$$

is Fin-convergent to x. If f is $\mathcal{I}^*$-convergent to x, then we write $\mathcal{I}^*\text{-}\lim f = x$.

The usual notion of $\mathcal{I}^*$-convergence of sequences is a special case for $S = \mathbb{N}$. Similarly as for the $\mathcal{I}$-convergence of sequences, we write $\mathcal{I}^*\text{-}\lim x_n = x$.

In fact, the $\mathcal{I}^*$-convergence was defined in [22] in a slightly different way – the Fin-convergence of the restriction $g|_M$ was used. It is easy to see that these two definitions are equivalent. Our approach will prove advantageous when using more complicated ideals instead of Fin.

In the definition of $\mathcal{I}^{\mathcal{K}}$-convergence we simply replace the ideal Fin by an arbitrary ideal on the set S.

Definition 3.2. Let $\mathcal{K}$ and $\mathcal{I}$ be ideals on a set S, let X be a topological space and let x be an element of X. The function $f\colon S \to X$ is said to be *$\mathcal{I}^{\mathcal{K}}$-convergent* to x if there exists a set $M \in \mathcal{F}(\mathcal{I})$ such that the function $g\colon S \to X$ given by

$$g(s) = \begin{cases} f(s), & \text{if } s \in M \\ x, & \text{if } s \notin M \end{cases}$$

is $\mathcal{K}$-convergent to x. If f is $\mathcal{I}^{\mathcal{K}}$-convergent to x, then we write $\mathcal{I}^{\mathcal{K}}\text{-}\lim f = x$.

As usual, in the case $S = \mathbb{N}$ we speak about $\mathcal{I}^{\mathcal{K}}$-convergence of sequences and use the notation $\mathcal{I}^{\mathcal{K}}\text{-}\lim x_n = x$.

Remark 3.3. The definition of $\mathcal{I}^{\mathcal{K}}$-convergence can be reformulated in the form of decomposition theorem. A function f is $\mathcal{I}^{\mathcal{K}}$-convergent if and only if it can be written as $f = g + h$, where g is $\mathcal{K}$-convergent and h is non-zero only on a set from $\mathcal{I}$. An analogous observation was made in [7] for the statistical convergence of sequences and in [31] for the statistical convergence of double sequences.

Remark 3.4. A definition of $\mathcal{I}^{\mathcal{K}}$-convergence following more closely the approach from [22] would be: there exists $M \in \mathcal{F}(\mathcal{I})$ such that the function $f|_M$ is $\mathcal{K}|M$-convergent to x where $\mathcal{K}|M = \{A \cap M; A \in \mathcal{K}\}$ is the trace of $\mathcal{K}$ on M. These two definitions are equivalent but the one given in Definition 3.2 is somewhat simpler.

One can show easily directly from the definitions that $\mathcal{K}$-convergence implies $\mathcal{I}^{\mathcal{K}}$-convergence.

Lemma 3.5. *If $\mathcal{I}$ and $\mathcal{K}$ are ideals on a set S and $f\colon S \to X$ is a function such that $\mathcal{K}\text{-}\lim f = x$, then $\mathcal{I}^{\mathcal{K}}\text{-}\lim f = x$.*

Using Lemma 2.2 (ii) and the definition of $\mathcal{I}^{\mathcal{K}}$-convergence we get immediately

Proposition 3.6. *Let $\mathcal{I}$, $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{K}$, $\mathcal{K}_1$ and $\mathcal{K}_2$ be ideals on a set S such that $\mathcal{I}_1 \subseteq \mathcal{I}_2$ and $\mathcal{K}_1 \subseteq \mathcal{K}_2$ and let X be a topological space. Then for any function $f\colon S \to X$ we have*

$$\mathcal{I}_1^{\mathcal{K}}\text{-}\lim f = x \qquad \Rightarrow \qquad \mathcal{I}_2^{\mathcal{K}}\text{-}\lim f = x,$$
$$\mathcal{I}^{\mathcal{K}_1}\text{-}\lim f = x \qquad \Rightarrow \qquad \mathcal{I}^{\mathcal{K}_2}\text{-}\lim f = x.$$

In what follows we are going to study the relationship between the $\mathcal{I}$-convergence and $\mathcal{I}^{\mathcal{K}}$-convergence. In particular, we will specify the conditions under which the implications

$$\mathcal{I}^{\mathcal{K}}\text{-}\lim f = x \qquad \Rightarrow \qquad \mathcal{I}\text{-}\lim f = x, \tag{3.1}$$
$$\mathcal{I}\text{-}\lim f = x \qquad \Rightarrow \qquad \mathcal{I}^{\mathcal{K}}\text{-}\lim f = x, \tag{3.2}$$

hold.

We start with the easier implication (3.1). In the case $\mathcal{K} = \mathrm{Fin}$ this implication is known to be true for the admissible ideals, that is, for ideals fulfilling $\mathcal{K} \subseteq \mathcal{I}$. We next show that the same is true in general.

Proposition 3.7. *Let $\mathcal{I}, \mathcal{K}$ be ideals on a set S, let X be a topological space and let f be a function from S to X.*

(i) *If the implication* (3.1) *holds for some point $x \in X$ which has at least one neighborhood different from X, then $\mathcal{K} \subseteq \mathcal{I}$. Consequently, if the implication* (3.1) *holds in a topological space that is not indiscrete, then $\mathcal{K} \subseteq \mathcal{I}$.*

(ii) *If $\mathcal{K} \subseteq \mathcal{I}$, then the implication* (3.1) *holds.*

Proof. (i) Suppose that $\mathcal{K} \nsubseteq \mathcal{I}$, that is, there exists a set $A \in \mathcal{K} \setminus \mathcal{I}$. Let x be a point with a neighborhood $U \subsetneqq X$ and $y \in X \setminus U$. Let us define a function $f\colon S \to X$ by

$$f(t) = \begin{cases} x & \text{if } t \notin A, \\ y & \text{otherwise.} \end{cases}$$

Clearly, $\mathcal{K}\text{-}\lim f = x$ and thus by Lemma 3.5 we get $\mathcal{I}^{\mathcal{K}}\text{-}\lim f = x$. As $f^{-1}(X \setminus U) = A \notin \mathcal{I}$, the function f is not $\mathcal{I}$-convergent to x

(ii) Let X be any topological space, $x \in X$ and $f\colon S \to X$. Let $\mathcal{K} \subseteq \mathcal{I}$ and $\mathcal{I}^{\mathcal{K}}\text{-}\lim f = x$. By the definition of $\mathcal{I}^{\mathcal{K}}$-convergence there exists $M \in \mathcal{F}(\mathcal{I})$ such that

$$C := f^{-1}(X \setminus U) \cap M \in \mathcal{K} \subseteq \mathcal{I}$$

for each neighborhood U of the point x. Consequently,

$$f^{-1}(X \setminus U) \subseteq (X \setminus M) \cup C \in \mathcal{I}$$

and thus $\mathcal{I}\text{-}\lim f = x$. □

3.2. **Additive property and $\mathcal{I}^{\mathcal{K}}$-convergence.** Inspired by [24] and [28] where the case $\mathcal{K} = \mathrm{Fin}$ and $S = \mathbb{N}$ is investigated, we now concentrate on an algebraic characterization of the ideals $\mathcal{I}$ and $\mathcal{K}$ such that the implication (3.2) holds for each function $f\colon S \to X$. Before doing this we need to prove some auxiliary results.

Definition 3.8. Let $\mathcal{K}$ be an ideal on a set S. We write $A \subset_{\mathcal{K}} B$ whenever $A \setminus B \in \mathcal{K}$. If $A \subset_{\mathcal{K}} B$ and $B \subset_{\mathcal{K}} A$, then we write $A \sim_{\mathcal{K}} B$. Clearly,

$$A \sim_{\mathcal{K}} B \qquad \Leftrightarrow \qquad A \triangle B \in \mathcal{K}.$$

We say that a set A is $\mathcal{K}$-*pseudointersection* of a system $\{A_n; n \in \mathbb{N}\}$ if $A \subset_{\mathcal{K}} A_n$ holds for each $n \in \mathbb{N}$.

In the case $\mathcal{K} = \mathrm{Fin}$ we obtain the notion of pseudointersection and the relations $\subseteq^*$ and $=^*$ which are often used in set theory (see [20, p.102]).

It is easy to see that using the symbols $\subset_{\mathcal{K}}$ and $\sim_{\mathcal{K}}$ can be understood as another way of speaking about the equivalence classes of the subsets of S in the quotient Boolean algebra $\mathcal{P}(S)/\mathcal{K}$.

In the following lemma we describe several equivalent formulations of a condition for ideals $\mathcal{I}$ and $\mathcal{K}$ which will play crucial role in further study.

Lemma 3.9. *Let $\mathcal{I}$ and $\mathcal{K}$ be ideals on the same set S. The following conditions are equivalent:*

(i) *For every sequence $(A_n)_{n\in\mathbb{N}}$ of sets from $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \subset_{\mathcal{K}} A$ for all n's.*
(ii) *Any sequence $(F_n)_{n\in\mathbb{N}}$ of sets from $\mathcal{F}(\mathcal{I})$ has a $\mathcal{K}$-pseudointersection in $\mathcal{F}(\mathcal{I})$.*
(iii) *For every sequence $(A_n)_{n\in\mathbb{N}}$ of sets belonging to $\mathcal{I}$ there exists a sequence $(B_n)_{n\in\mathbb{N}}$ of sets from $\mathcal{I}$ such that $A_j \sim_{\mathcal{K}} B_j$ for $j \in \mathbb{N}$ and $B = \bigcup_{j\in\mathbb{N}} B_j \in \mathcal{I}$.*
(iv) *For every sequence of mutually disjoint sets $(A_n)_{n\in\mathbb{N}}$ belonging to $\mathcal{I}$ there exists a sequence $(B_n)_{n\in\mathbb{N}}$ of sets belonging to $\mathcal{I}$ such that $A_j \sim_{\mathcal{K}} B_j$ for $j \in \mathbb{N}$ and $B = \bigcup_{j\in\mathbb{N}} B_j \in \mathcal{I}$.*
(v) *For every non-decreasing sequence $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \dots$ of sets from $\mathcal{I}$ there exists a sequence $(B_n)_{n\in\mathbb{N}}$ of sets belonging to $\mathcal{I}$ such that $A_j \sim_{\mathcal{K}} B_j$ for $j \in \mathbb{N}$ and $B = \bigcup_{j\in\mathbb{N}} B_j \in \mathcal{I}$.*
(vi) *In the Boolean algebra $\mathcal{P}(S)/\mathcal{K}$ the ideal $\mathcal{I}$ corresponds to a σ-directed subset, that is, every countable subset has an upper bound.*

Note that (ii) is just a dual formulation of (i). Similarly, (vi) is the formulation of (i) in the language of Boolean algebras. The equivalence of (iii), (iv), (v) can be easily shown by the standard methods from the measure theory. Proof of the equivalence of the remaining conditions is similar to the proof of Proposition 1 of [3], where the case $\mathcal{K} = \text{Fin}$ is considered. We include the proof for the sake of completeness and also to stress that the validity of this lemma does not depend on the countability of S or the assumption that $\mathcal{K} \subseteq \mathcal{I}$.

Proof. (i)$\Rightarrow$ (v) Let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \dots$ be a non-decreasing sequence of sets from $\mathcal{I}$. Since each $A_n \in \mathcal{I}$, the condition (i) yields the existence of a set $A \in \mathcal{I}$ satisfying $A_n \subset_{\mathcal{K}} A$ for $n \in \mathbb{N}$. Let $B_n := A \cap A_n$. Since $B_n \subseteq A$, we have $B_n \in \mathcal{I}$. Moreover, $B_n \triangle A_n = A_n \setminus A \in \mathcal{K}$, thus $B_n \sim_{\mathcal{K}} A_n$. Finally, $B = \bigcup_{j\in\mathbb{N}} B_j \subseteq A \in \mathcal{I}$, as required.

(iii)$\Rightarrow$ (i) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets belonging to $\mathcal{I}$. By (iii) there exists a sequence $(B_n)_{n\in\mathbb{N}}$ of sets from $\mathcal{I}$ such that for all n we have $B_n \sim_{\mathcal{K}} A_n$ and $A := \bigcup_{n\in\mathbb{N}} B_n \in \mathcal{I}$. From $A_n \triangle B_n \in \mathcal{K}$ and $B_n \subseteq A$ we get $A_n \subset_{\mathcal{K}} A$, which proves (i). □

It is also easy to see that in condition (ii) it suffices to consider only sequences of sets from a filterbase. This reformulation of (ii) can be sometimes easier to prove.

Definition 3.10. Let $\mathcal{I}$, $\mathcal{K}$ be ideals on a set S. We say that $\mathcal{I}$ has the *additive property* with respect to $\mathcal{K}$, or more briefly that $\text{AP}(\mathcal{I}, \mathcal{K})$ holds, if any of the equivalent conditions of Lemma 3.9 holds.

The condition AP from [24], which characterizes ideals such that $\mathcal{I}^*$-convergence implies $\mathcal{I}$-convergence, is equivalent to the condition $\text{AP}(\mathcal{I}, \text{Fin})$. Let us note that ideals fulfilling this condition are often called *P-ideals* (see for example [3] or [14]).

In the following two theorems we show that the condition $\text{AP}(\mathcal{I}, \mathcal{K})$ is the correct generalization of conditions AP from [24], [28] and [8]. In particular, as special cases of our results we obtain Theorem 3.1 of [24], Theorem 8 of [29] and Theorem 2 of [8].

Although we do not consider arbitrary topological spaces, we feel that the restriction to the first countable spaces is sufficient for most applications. For example, in [24] the authors work only with metric spaces and in [28] the case that X is a first countable T_1-space is considered.

Theorem 3.11. *Let $\mathcal{I}$ and $\mathcal{K}$ be ideals on a set S and let X be a first countable topological space. If the ideal $\mathcal{I}$ has the additive property with respect to $\mathcal{K}$, then for any function $f \colon S \to X$ the implication* (3.2) *holds. In other words, if the condition* $\text{AP}(\mathcal{I}, \mathcal{K})$ *holds, then the $\mathcal{I}$-convergence implies the $\mathcal{I}^{\mathcal{K}}$-convergence.*

Proof. Let $f \colon S \to X$ be an $\mathcal{I}$-convergent function and let $x = \mathcal{I}\text{-}\lim f$. Let $\mathcal{B} = \{U_n; n \in \mathbb{N}\}$ be a countable base for X at the point x. By the $\mathcal{I}$-convergence of f we have

$$f^{-1}(U_n) \in \mathcal{F}(\mathcal{I})$$

for each n, thus by Lemma 3.9 there exists $A \in \mathcal{F}(\mathcal{I})$ with $A \subset_{\mathcal{K}} f^{-1}(U_n)$, that is, $A \setminus f^{-1}(U_n) \in \mathcal{K}$ for all n's.

Now it suffices to show that the function $g\colon S \to X$ given by $g|_A = f|_A$ and $g[S \setminus A] = \{x\}$ is $\mathcal{K}$-convergent to x. As for $U_n \in \mathcal{B}$ we have

$$g^{-1}(U_n) = (S \setminus A) \cup f^{-1}(U_n) = S \setminus (A \setminus f^{-1}(U_n)),$$

and the set $A \setminus f^{-1}(U_n)$ belongs to $\mathcal{K}$, its complement $g^{-1}(U_n)$ lies in $\mathcal{F}(\mathcal{K})$, as required. □

Let us recall that a topological space X is called *finitely generated space* or *Alexandroff space* if any intersection of open subsets of X is again an open set (see [1]). Equivalently, X is finitely generated if and only if each point of x has a smallest neighborhood. Finitely generated T_1-spaces are precisely the discrete spaces.

Theorem 3.12. *Let $\mathcal{I}$, $\mathcal{K}$ be ideals on a set S and let X be a first countable topological space which is not finitely generated. If the implication* (3.2) *holds for any function $f\colon S \to X$, then the ideal $\mathcal{I}$ has the additive property with respect to $\mathcal{K}$.*

Proof. Let $x \in X$ be an accumulation point of X which does not have a smallest neighborhood. Let $\mathcal{B} = \{U_i; i \in \mathbb{N} \cup \{0\}\}$ be a countable base at x such that $U_n \supsetneq U_{n+1}$ and $U_0 = X$. Suppose we are given some countable family A_n of mutually disjoint sets from $\mathcal{I}$.

For each $n \in \mathbb{N}$ choose an $x_n \in U_{n-1} \setminus U_n$. Let us define $f\colon S \to X$ as

$$f(s) = \begin{cases} x_n & \text{if } s \in A_n, \\ x & \text{if } s \notin \bigcup_{n\in\mathbb{N}} A_n. \end{cases}$$

We have $f^{-1}(X \setminus U_n) = \bigcup_{i=1}^{n} A_i \in \mathcal{I}$, hence $\mathcal{I}\text{-}\lim f = x$. By the assumption, $\mathcal{I}^{\mathcal{K}}\text{-}\lim f = x$, which means that there is $A \in \mathcal{F}(\mathcal{I})$ such that the function $g\colon S \to X$ given by $g|_A = f|_A$ and $g[S \setminus A] = \{x\}$ is $\mathcal{K}$-convergent to x. This yields

$$g^{-1}(X \setminus U_n) = \left(\bigcup_{i=1}^{n} A_i\right) \cap A = \bigcup_{i=1}^{n} (A_i \cap A) \in \mathcal{K}.$$

From this we have $A_i \cap A \in \mathcal{K}$, thus $B_i := A_i \setminus A \sim_{\mathcal{K}} A_i$.

Note that, at the same time

$$\bigcup_{i\in\mathbb{N}} B_i = \left(\bigcup_{i\in\mathbb{N}} A_i\right) \setminus A \subseteq S \setminus A \in \mathcal{I}.$$

We have shown (iv) from Lemma 3.9. □

Remark 3.13. Let us note that we have in fact proved a slightly stronger result: Whenever x is an accumulation point of X such that there exists a countable basis at x, the point x does not have a smallest neighborhood and the implication (3.2) holds for each function $f\colon S \to X$ which is $\mathcal{I}^{\mathcal{K}}$-convergent to x, then the ideal $\mathcal{I}$ has the additive property with respect to $\mathcal{K}$.

We next provide an example showing that Theorem 3.11 does not hold in general for spaces which are not first countable.

Example 3.14. Pointwise $\mathcal{I}$-convergence of sequences of continuous real functions was studied in [18] and [19]. It can be understood as convergence of sequences of elements of the space $C_p(X)$ of all real continuous function endowed with the topology of pointwise convergence. The authors of [18, 19] defined and studied the $\mathcal{I}$-convergence property which, using our terminology, can be formulated as follows: A topological space X has the *$\mathcal{I}$-convergence property* if (3.2) holds in the space $C_p(X)$ for $S = \mathbb{N}$ and $\mathcal{K} = \mathrm{Fin}$.

It is known that $C_p(X)$ is first countable if and only if X is countable, see [30, Theorem 4.4.2]. Hence our Theorem 3.11 yields that all countable spaces have the $\mathcal{I}$-convergence property for every P-ideal $\mathcal{I}$. The same result was obtained in [18, Corollary 1].

It was shown in [19] that $\mathbb{R}$ does not have $\mathcal{I}$-convergence property for any nontrivial analytic P-ideal on $\mathbb{N}$. (By trivial ideals we mean the ideals of the form $\mathcal{I}_C = \{A \subseteq \mathbb{N}; A \subseteq^* C\}$ for some $C \subseteq \mathbb{N}$.) Hence, $C_p(\mathbb{R})$ provides the desired counterexample, which works for a large class of ideals on $\mathbb{N}$. The definition of analytic ideals, more related results and many examples of analytic P-ideals can be found, for example, in [13, 14].

To find a counterexample showing that Theorem 3.12 is in general not true without the assumption that the space X is first countable we can use any space in which all $\mathcal{I}$-convergent sequences are, in some sense, trivial.

Example 3.15. Let us recall that ω_1 denotes the first uncountable ordinal with the usual ordering. Let X be the topological space on the set $\omega_1 \cup \{\omega_1\}$ with the topology such that all points different from ω_1 are isolated and the base at the point ω_1 consists of all sets $U_\alpha = \{\beta \in X; \beta > \alpha\}$ for $\alpha < \omega_1$. Notice that if $C \subseteq \omega_1$ is a set such that $\omega_1 \in \overline{C}$, then $|C| = \aleph_1$.

Now let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$ and let a function $f\colon \mathbb{N} \to X$ be $\mathcal{I}$-convergent to ω_1. We will show that then there exists $M \in \mathcal{F}(\mathcal{I})$ such that $f(x) = \omega_1$ for each $x \in M$, that is, $f|_M$ is constant. Clearly, this implies that f is $\mathcal{I}^*$-convergent.

For the sake of contradiction, suppose that each set $M \in \mathcal{F}(\mathcal{I})$ contains some point m such that $f(m) \neq \omega_1$. Since $f^{-1}(U) \in \mathcal{F}(\mathcal{I})$, for any neighborhood U of ω_1 in X there exists $m \in \mathbb{N}$ with $f(m) \in U \setminus \{\omega_1\}$. Therefore for the set $C = \{m \in \mathbb{N}; f(m) \neq \omega_1\}$ we have $\omega_1 \in \overline{f[C]}$. Since $f[C] \subseteq \omega_1$ and it is a countable set contained in ω_1, this is a contradiction.

Now, by choosing an ideal $\mathcal{I}$ which does not have the additive property $\mathrm{AP}(\mathcal{I}, \mathrm{Fin})$ we obtain the desired counterexample.

4. Examples and applications

We have already mentioned that our motivation for definition and study of $\mathcal{I}^{\mathcal{K}}$-convergence was an effort to provide a common generalization to the notion of $\mathcal{I}^*$-convergence which was defined first for the usual sequences in [22] and later generalized for sequences of functions, double sequences and nets in [16], [25] and [29], respectively.

In this section we show that the notion of the $\mathcal{I}^{\mathcal{K}}$-convergence is a correct generalization of these notions, that is, all these notions are special cases of the $\mathcal{I}^{\mathcal{K}}$-convergence. We begin with the notion of $\mathcal{I}^*$-convergence of double sequences.

4.1. **Double sequences.** In the study of double sequences several types of convergence are used. For our purposes, the following one is the most important.

Definition 4.1 ([2, 33])**.** A double sequence $(x_{m,n})_{m,n=1}^{\infty}$ of points of a topological space X is said to converge to x *in Pringsheim's sense* if for each neighborhood U of the point x

$$(\exists k \in \mathbb{N})(\forall m \geq k)(\forall n \geq k) x_{m,n} \in U.$$

It is easy to see that the convergence in Pringsheim's sense is equal to the $\mathcal{I}$-convergence along the *Pringsheim's ideal* $\mathcal{I}_2$ on $\mathbb{N} \times \mathbb{N}$ whose dual filter $\mathcal{F}(\mathcal{I}_2)$ is given by the filterbase

$$\mathcal{B}_2 = \{[m, \infty) \times [m, \infty); m \in \mathbb{N}\}.$$

We will give a different description of this ideal in Example 4.2.

Altogether four types of convergence of double sequences were studied in [2]. All of them can be described as $\mathcal{I}$-convergences using appropriate ideals on $\mathbb{N} \times \mathbb{N}$ (see Figure 1). In fact, we denote the Pringsheim's ideal by $\mathcal{I}_2$ in order to be consistent with the notation of [2].

The $\mathcal{I}^*$-convergence of double sequences studied in [25] and [8] is the same as $\mathcal{I}^{\mathcal{I}_2}$-convergence in $\mathbb{N} \times \mathbb{N}$. Therefore, as a special case of our Theorems 3.11 and 3.12 for $S = \mathbb{N} \times \mathbb{N}$ and $\mathcal{K} = \mathcal{I}_2$ we obtain Proposition 4.2 of [25], and Theorems 3 and 4 of [8]. Note that in [25] and [8] only the ideals containing $\mathcal{I}_2$ are considered, see Proposition 3.7.

4.2. **Further examples.** In order to avoid technical details we will define neither the notions of pointwise and uniform $\mathcal{I}^*$-convergence of a sequence of functions defined in [16], nor the notions of the $\mathcal{I}$- and $\mathcal{I}^*$-convergence of nets defined in [29].

We just mention that, given an ideal $\mathcal{L}$ on $\mathbb{N}$, the uniform $\mathcal{L}^*$-convergence of a sequence of functions defined on X is precisely the $\mathcal{I}^{\mathcal{K}}$-convergence for the ideal $\mathcal{I}$ on $X \times \mathbb{N}$ given by the

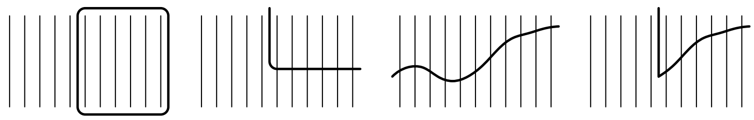

FIGURE 1. Ideals from [2] illustrated by depicting typical sets from the filterbase. Vertical lines represent the partition of $\mathbb{N}\times\mathbb{N}$ into countably many infinite sets $\{i\}\times\mathbb{N}$.

filterbase $\{X\times(\mathbb{N}\setminus A); A\in\mathcal{L}\}$ and the ideal $\mathcal{K}$ given by the filterbase $\{X\times(\mathbb{N}\setminus A); A\in \mathrm{Fin}\}$. The pointwise $\mathcal{L}^*$-convergence can be obtained if $\mathcal{I}$ is the ideal of all sets $A\subseteq X\times\mathbb{N}$ such that for each $x\in X$ the x-cut $A_x:=\{n\in\mathbb{N}; (x,n)\in A\}$ belongs to $\mathcal{L}$, and $\mathcal{K}$ consists of all sets such that each A_x is finite.

In both cases it can be shown that the condition $\mathrm{AP}(\mathcal{I},\mathcal{K})$ is equivalent to the condition $\mathrm{AP}(\mathcal{L},\mathrm{Fin})$. Hence our Theorems 3.11 and 3.12 imply that these two types of $\mathcal{I}$-convergence are equivalent to corresponding $\mathcal{I}^*$-convergence if and only if $\mathrm{AP}(\mathcal{L},\mathrm{Fin})$ holds. This observation has been made already in [16].

Similarly, the concept of $\mathcal{I}^*$-convergence of nets is a special case of $\mathcal{I}^{\mathcal{K}}$-convergence and Theorem 12 of [29] can be obtained from our Theorems 3.11 and 3.12 by choosing the section filter of the considered directed set for $\mathcal{K}$ (the definition of the section filter can be found, for example, in [5, p.60]).

4.3. **$\mathcal{I}$-convergence of double sequences.** We close this paper with an observation concerning the $\mathcal{I}$-convergence of double sequences.

Notice that any bijection between sets S and T naturally gives rise to a bijection between X^S and X^T, an isomorphism between Boolean algebras $\mathcal{P}(S)$ and $\mathcal{P}(T)$ and also to an isometric isomorphism between linear normed spaces $\ell_\infty(S)$ and $\ell_\infty(T)$. It is easy to see that this correspondence preserves also the properties related to the notion of $\mathcal{I}$-convergence. Hence results about $\mathcal{I}$-convergence for a given set S do not depend on the natural (partial) ordering on the set S in any way. Thus these results can be transferred to any set of the same cardinality.

We can use any bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$ to relate results about sequences and double sequences. It is interesting to note that several authors working in this area did not realize this possibility.

The basic results on $\mathcal{I}$-convergence (such as additivity, multiplicativity, uniqueness of limit in Hausdorff spaces) need not be shown again for double sequences, since they follow from the analogous result for sequences; although the proofs are rather trivial in both cases. But there are also some more interesting concepts that were defined for double sequences in a such way that they are preserved by this correspondence. Namely, this is true for the notions of $\mathcal{I}$-Cauchy double sequences, extremal $\mathcal{I}$-limit points ($\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior) and $\mathcal{I}$-cluster points.

In this way, some results from the papers [9, 17, 26, 36] on the above mentioned concepts can be obtained from the results of [10, 11, 23, 27]. Actually, the fact that a double sequence is $\mathcal{I}$-convergent if and only if it is $\mathcal{I}$-Cauchy is shown in Proposition 5 of [11] using a bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$.

The above observation can also be used to get an alternative description the ideal $\mathcal{I}_2$.

Example 4.2. A basic example of an ideal which does not have the property $\mathrm{AP}(\mathcal{I},\mathrm{Fin})$ is the ideal $\mathcal{I}_m$ given in Example 1.1.(g) of [24] and Example (XI) of [22]. It is defined as follows: Suppose we are given any partition $\mathbb{N}=\bigcup_{i=1}^n D_i$ of $\mathbb{N}$ into countably many infinite sets. A set $A\subseteq\mathbb{N}$ belongs to $\mathcal{I}_m$ if and only if it intersects only finitely many D_i's. Of course, choosing different partitions of $\mathbb{N}$ can lead to ideals which are different, but equivalent from the point of view of $\mathcal{I}$-convergence.

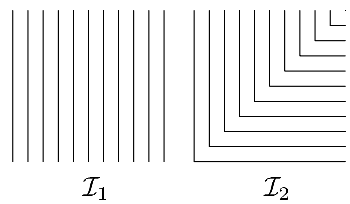

FIGURE 2. Partitions of $\mathbb{N}\times\mathbb{N}$ defining the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$

We can also use any countable set instead of $\mathbb{N}$. In particular, as observed in the proof of Corollary 4 in [2], by choosing the partition of $\mathbb{N}\times\mathbb{N}$ into sets $D_i=\{(n,i);n\geq i\}\cup\{(i,k);k\geq i\}$ we obtain the ideal $\mathcal{I}_2$ in this way. Similarly, by using $D_i=\{i\}\times\mathbb{N}$ we get the ideal $\mathcal{I}_1$ of [2] (see Figure 2). Thus the ideals $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_m$ are essentially the same. In particular, this gives an alternative proof that $\mathrm{AP}(\mathcal{I}_2,\mathrm{Fin})$ and $\mathrm{AP}(\mathcal{I}_1,\mathrm{Fin})$ fail, see [8].

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