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HOMOMORPHIC EXTENSIONS OF PSEUDOCOMPLEMENTED SEMILATTICES

TIBOR KATRIŇÁK AND JAROSLAV GURIČAN

Dedicated to the 70th birthday of Alfonz Haviar

Abstract. Our aim is to study and characterize extensions to a homomorphism in the class of pseudocomplemented semilattices. We present here such a description.

1. Introduction

We shall deal with the question in which circumstances a mapping f defined on a set X of generators of a pseudocomplemented semilattice S can be extended to a homomorphism $g : S \to M$. Such an extension, if it exists, is uniquely determined.

It is a well-known fact (see [5]) that the class of all pseudocomplemented semilattices is equational with only one non-trivial subvariety, namely, the class of Boolean algebras. The preceding question found an answer for Boolean algebras (see [9] and especially Sikorski's extension criterion). We shall use these results as a motivation for our task.

2. Preliminaries

A pseudocomplemented semilattice (= PCS) is an algebra $(S; \wedge, ^*, 0, 1)$ of type (2,1,0,0), where $(S; \wedge, 0, 1)$ is a bounded meet-semilattice and, for every $a \in S$, the element a^* is a *pseudocomplement* of a, i.e. $x \leq a^*$ if and only if $x \wedge a = 0$. A PCS S is said to be *non-trivial*, whenever $\mid S \mid \geq 2$. An element $a \in S$ is called *closed*, if $a = a^{**}$. Let $B(S)$ denote the set of all closed elements of S. It is known that

$$(B(S); +, \wedge, ^*, 0, 1)$$

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forms a Boolean algebra with

$$a + b = (a^* \wedge b^*)^*$$

(see [1] and [3]). (Clearly, a PCS S is a Boolean algebra if and only if S satisfies the identity $x = x^{**}$.)

Here are some rules of computation with * and $\wedge$ (see [1] or [3]):

(1) $x \wedge x^* = 0$.

(2) $x \leq y$ implies that $x^* \geq y^*$.

(3) $x \leq x^{**}$.

(4) $x^* = x^{***}$.

(5) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$.

(6) $0^* = 1$ and $1^* = 0$.

The following result can be easily verified (see [7]).

Lemma 2.1. *Let S be a PCS and let $X \subseteq S$. Then S is generated by X, i.e. $S = [X]$ if and only if $[X^{**}]_{Bool} = B(S)$ and $S = [X \cup B(S)]_{sem}$, that means, $B(S)$ is generated by $X^{**} = \{x^{**} : x \in X\}$ as a Boolean algebra and S is generated by $X \cup B(S)$ as a semilattice.*

Let S and T be PCS's. A function $f : S \to T$ is called a *homomorphism* (of PCS's) if $f(x \wedge y) = f(x) \wedge f(y)$, $f(x)^* = f(x^*)$ for $x, y \in S$. We observe that $f(0) = 0$, and $f(1) = 1$.

The definitions of the concepts discussed in this paper may be found in [1] and [3].

3. Extensions

Let S and K be PCS's and let K be a subalgebra of S, that means, S is an *extension* of K. (Notation: $K \leq S$.) In addition, we set $K[X] = [K \cup X]$, whenever $X \subseteq S$. We say that S is a *finite* (*simple*) extension of its subalgebra K, if $S = K[X]$ for some finite (one-element) set $X \subseteq S$.

Proposition 3.1. *Let K and S be PCS's. Then S is a simple extension of K, that means, $S = K[x]$ for some $x \in S$, if and only if*

(i) $B(S) = [B(K) \cup \{x^{**}\}]_{Bool}$,

(ii) $S_1 = [B(S) \cup K]_{sem}$ *is a subalgebra of S and*

(iii) $S = [S_1 \cup \{x\}]_{sem}$.

Proof. Assume first $S = K[x]$. Then (i) is straightforward (see Lemma 2.1). (ii) We have only to show that $u \in S_1$ implies $u^* \in S_1$. Really, take $u \in S_1$. Evidently,

$$u = a \wedge s$$

for some $a \in B(S)$ and $s \in K$. Since $u \in S$, we have

$$u^{**} = (a \wedge s)^{**} = a^{**} \wedge s^{**} = a \wedge s^{**} \in B(S).$$

Therefore, $u^* = u^{***} = (a \wedge s^{**})^* \in B(S) \subseteq S_1$, and S_1 is a subalgebra of S.

(iii) Set $M = [S_1 \cup \{x\}]_{sem}$. We claim that M is a subalgebra of S. Similarly as above, we have only to show that $u \in M$ implies $u^* \in M$. Either $u \in S_1$ or there exists $s \in S_1$ such that

$$u = s \wedge x.$$

In the first event $u^* \in S_1$. In the second one, we get $u^{**} = (s \wedge x)^{**} \in B(S)$. Since $B(S_1) = B(S)$, it is easy to see that $u^* \in S_1$, and hence M is a subalgebra of S. Finally, since $K \cup \{x\} \subseteq M$, we obtain $M = S$.

To prove the converse, assume that the conditions (i)-(iii) are satisfied. It is easy to see that $K \leq S$. Therefore, $K[x] = [K \cup \{x\}] \subseteq S$. On the other hand, $B(S) \subseteq K[x]$ by (i). Consequently, $S \subseteq K[x]$ by (ii) and (iii), and the proof is complete.

Proposition 3.1 generalizes immediately to arbitrary set X (instead of one-element set $\{x\}$).

Theorem 3.2. *Let K and S be PCS's. Then $S = K[X]$ for some $X \subseteq S$ if and only if*

(i) $B(S) = [B(K) \cup X^{**}]_{bool}$,
(ii) $S_1 = [B(S) \cup K]_{sem}$ *is a subalgebra of S and*
(iii) $S = [S_1 \cup X]_{sem}$.

Corollary 3.3. *Let $S = K[X]$ and let $u \in S$. Then there exist $s \in K$ and a finite $U \subseteq X$ such that*

$$u = u^{**} \wedge s \wedge \bigwedge(x : x \in U).$$

For our next result we need the following concept:

Definition 3.4. *Let K and S be bounded meet-semilattices* (PCS's) *such that $K \leq S$. Then K is said to be* relatively complete *in S, if for each $b \in S$ there exists a smallest $a \in K$ such that $b \leq a$. In notation:*

$$a = \Pr(b) = \Pr_K^S(b) = \min\{x \in K \mid b \leq x\}.$$

Write $K \leq_{rc} S$ if K is relatively complete in S. See also [6] or [9] for relatively complete lattices or Boolean algebras.

Using the notation from the preceding theorem, we can formulate the following result:

Corollary 3.5. *Let $K \leq S$ for PCS's. Then $K \leq_{rc} S$ if and only if*

$$K \leq_{rc} S_1 \leq_{rc} S,$$

where $S_1 = [B(S) \cup K]_{sem}$.

Proof. Let $K \leq_{rc} S$. (Clearly, $S = K[X]$ for some $X \subseteq S$.) It follows that $B(K) \leq_{rc} B(S)$ and $K \leq_{rc} S_1$. It remains to prove $S_1 \leq_{rc} S$. Let $u \in S$ and

$u \leq v$ for some $v \in S_1$. It is easy to see that $v = a \wedge t$ for some $a \in B(S)$ and $t \in K$. Now, $u \leq v$ if and only if $u \leq a$ and $u \leq t$ in S. But $u \leq a$ if and only if $u^{**} \leq a$. The second relation $u \leq t$ is equivalent to $u \leq \mathrm{Pr}_K^S(u) \leq t$. Therefore,

$$u \leq u^{**} \wedge \mathrm{Pr}_K^S(u) \leq a \wedge t = v.$$

Since $u^{**} \wedge \mathrm{Pr}_K^S(u) \in S_1$, we have $S_1 \leq_{rc} S$. The converse implication is straightforward.

4. Extension to a homomorphism

In this section we shall examine the following situation: Let K, M and $S = K[X]$ be PCS's. Let $f_0 : K \to M$ be a homomorphism and $f : X \to M$ be a mapping. The question concerning f is whether or not there exists a homomorphism $g : S \to M$ such that $g \restriction_{K \cup X} = f_0 \cup f$ (= the restriction of g to $K \cup X$). It is easy to see that g, whenever it exists, is uniquely determined. In this case we say that g is an *extension of* $f_0 \cup f$ *to a homomorphism.*

Notice that a specialization of our question for Boolean algebras has been considered by R. Sikorski. He found a useful characterization of those mappings f, for which there exists an extension to a Boolean homomorphism (see Sikorski's extension criterion in [9]).

The next theorem is concerned with a more general situation and will frequently be useful:

Theorem 4.1. *Let K, M and S be PCS's and let S be an extension of K, that means, $S = K[X]$ for some $X \subseteq S$. Assume that $f_0 : K \to M$ is a homomorphism and let $f : X \to M$ be a mapping. Then there exists a homomorphism $g : S \to M$ extending $f_0 \cup f$ if and only if the following conditions are fulfilled:*

- (i) *there is a Boolean homomorphism $h : B(S) \to B(M)$, which is an extension of $(f_0)_B : B(K) \to B(M)$ (i.e. $(f_0)_B$ is a restriction of f_0 to $B(K)$) such that*
$$h(x^{**}) = f(x)^{**}$$
for every $x \in X$;
- (ii) *if $S_1 = [B(S) \cup K]_{sem}$, then there exists a meet-semilattice homomorphism $f_1 : S_1 \to M$ such that f_1 is an extension of $f_0 \cup h$;*
- (iii) *there exists a meet-semilattice homomorphism $g : S \to M$ which is an extension of $f_1 \cup f$.*

In addition, the homomorphism $g : S \to M$, if it exists, is uniquely determined. If $u \in S$, then

$$g(u) = h(u^{**}) \wedge f_0(s) \wedge \bigwedge(f(x) : x \in U) = f_1(u^{**} \wedge s) \wedge \bigwedge(f(x) : x \in U)$$

for some $s \in K$ and a finite $U \subseteq X$ (see Corollary 3.3).

Proof. The necessity of (i)-(iii) is straightforward (see Lemma 2.1 and Theorem 3.2). Conversely, assume conditions (i) - (iii). First we show that $f_1 : S_1 \to M$ is a PCS-homomorphism. Really, suppose $u \in S_1$. By Theorem 3.2, $u = a \wedge s$ for some $a \in B(S)$ and $s \in K$. Therefore,

$$f_1(u) = f_1(a \wedge s) = h(a) \wedge f_0(s),$$

by (ii). Now,

$$\begin{aligned} f_1(u)^{**} &= (h(a) \wedge f_0(s))^{**} = h(a)^{**} \wedge f_0(s)^{**} = h(a) \wedge f_0(s^{**}) \\ &= h(a) \wedge h(s^{**}) = h(a \wedge s^{**}) = h(u^{**}) = f_1(u^{**}), \end{aligned}$$

by (i) and (ii). Hence,

$$f_1(u)^* = f_1(u)^{***} = h(u^{**})^* = h(u^*) = f_1(u^*),$$

as h is a Boolean homomorphism. Clearly, f_1 is a PCS's homomorphism and an extension of $f_0 \cup h$.

Now, we can show that meet-semilattice homomorphism $g : S \to M$ satisfies $g(u)^* = g(u^*)$ for any $u \in S$ as well. Really, take $u \in S$. By Theorem 3.2, either $u \in S_1$ or $u = s \wedge (\bigwedge X_1)$ for some $s \in S_1$ and a finite non-empty $X_1 \subseteq X$. The first case is straightforward: $g(u) = f_1(u)$. Let us consider the second event. By hypothesis,

$$g(u) = g(s \wedge (\bigwedge X_1)) = g(s) \wedge \bigwedge(g(y) : y \in X_1) = f_1(s) \wedge \bigwedge(g(y) : y \in X_1).$$

Since $g(y)^{**} = f(y)^{**} = f(y^{**}) = h(y^{**})$, for $y \in X_1$, we get

$$g(u)^{**} = f_1(s)^{**} \wedge \bigwedge(g(y)^{**} : y \in X_1) = h(s^{**}) \wedge h(\bigwedge X_1^{**}) = h(u^{**}).$$

It follows that

$$g(u)^* = g(u)^{***} = (g(u)^{**})^* = h(u^{**})^* = h(u^*) = f_1(u^*) = g(u^*),$$

by (i) - (iii). Now, it is easy to see that g is the required homomorphism extending $f_0 \cup f$. The last statement follows from Theorem 3.2 and Corollary 3.3. The proof is complete.

Corollary 4.2. *Under the assumptions of Theorem 4.1 and the additional hypothesis that $B(K) = B(S)$, the following statements are equivalent:*

(i) *There exists a PCS-homomorphism $g : S \to M$, which is an extension of $f_0 \cup f$.*

(ii) *There exists a meet-semilattice homomorphism $g : S \to M$, which is an extension of $f_0 \cup f$.*

Proof. Clearly, $B(K) = B(S)$ yields that $h \subseteq f_0$. Hence $f_1 = f_0$ and the rest follows from Theorem 4.1.

Theorem 4.1 shows that an extension of a PCS-homomorphism can be reduced to three special parts: one extension of a Boolean homomorphism and two extensions of bounded meet-semilattice homomorphisms. More precisely, let K, M and S be PCS's and let $S = K[X]$. Assume that there exist a PCS-homomorphism $f_0 : K \to M$ and a mapping $f : X \to M$. Then there exists

(I) a Boolean homomorphism $(f_0)_B : B(K) \to B(M)$, where $(f_0)_B$ is the restriction of f_0 to $B(K)$ (Lemma 2.1). In addition, there is a mapping $f^+ : X^{**} \to B(M)$ defined by the rule

$$f^+(x^{**}) = f(x)^{**}.$$

The first question concerning $(f_0)_B$ is whether or not there is an extension of $(f_0)_B \cup f^+$ to a Boolean homomorphism $h : B(S) \to B(M)$. (Notice that $[B(K) \cup X^{**}]_{Bool} = B(S)$ by Lemma 2.1.) The answer to this question comes from the following lemma, due to R. Sikorski (see [9], Theorem 5.5). First we need a new notation: Let B be a Boolean algebra. For $x \in B$ and $\varepsilon \in \{+1, -1\}$, define the element x^ε of B by

$$x^{+1} = x, \quad x^{-1} = x^*.$$

Lemma 4.3. *A Boolean homomorphism $h : B(S) \to B(M)$ is an extension of $(f_0)_B \cup f^+$ if and only if*

$$a^{\varepsilon_0} \wedge (x_1^{**})^{\varepsilon_1} \wedge \cdots \wedge (x_k^{**})^{\varepsilon_k} = 0$$

in $B(S)$ for $a \in B(K)$, $x_1^{**}, \cdots, x_k^{**} \in X^{**}$ and $\varepsilon_i \in \{+1, -1\}$ implies

$$f_0(a)^{\varepsilon_0} \wedge f(x_1^{**})^{\varepsilon_1} \wedge \cdots \wedge f(x_k^{**})^{\varepsilon_k} = 0$$

in $B(M)$.

(II) Suppose now that a Boolean homomorphism $h : B(S) \to B(M)$ exists and h is an extension of $(f_0)_B \cup f^+$. In addition, there exists $S_1 \leq S$ and we can ask again whether or not there exists a meet-semilattice homomorphism $f_1 : S_1 \to M$, which is an extension of $f_0 \cup h$. The answer can be formulated as follows:

Lemma 4.4. *Let $h : B(S) \to B(M)$ be a Boolean homomorphism and an extension of $(f_0)_B \cup f^+$. Then there exists a meet-semilattice homomorphism $f_1 : S_1 \to M$, which is an extension of $f_0 \cup h$ if and only if*

$$a \wedge s = b \wedge t$$

implies

$$h(a) \wedge f_0(s) = h(b) \wedge f_0(t)$$

for any $a, b \in B(S)$ and $s, t \in K$.

Proof. The result requires only routine verification, and the proof can be omitted.

(III) It remains to establish the third part. We thus have a semilattice homomorphism $f_1 : S_1 \to M$, which is an extension of $f_0 \cup h$. Since $S = [S_1 \cup X]_{sem}$ (Theorem 3.2), it is reasonable to ask again whether or not there exists a meet-semilattice homomorphism $g : S \to M$, which is an extension of $f_1 \cup f$. The following lemma yields a solution:

Lemma 4.5. *Let $f_1 : S_1 \to M$ be a semilattice homomorphism extending $f_0 \cup h$. Then there exists a semilattice homomorphism $g : S \to M$, which is an extension of $f_1 \cup f$ if and only if*

$$s \wedge \bigwedge(y : y \in Y) = t \wedge \bigwedge(z : z \in Z)$$

implies

$$f_1(s) \wedge \bigwedge(f(y) : y \in Y) = f_1(t) \wedge \bigwedge(f(z) : z \in Z)$$

for any $s, t \in S_1$ and arbitrary finite $Y, Z \subseteq X$.

The proof is routine.

We conclude this section by observing that Lemmas 4.3-4.5 complete Theorem 4.1. The interested reader should have no serious difficulty in reconstructing the corresponding theorem.

5. Simple extensions

In the last section (Theorem 4.1) we saw how a PCS-homomorphism $f_0 : K \to M$ can be extended to a PCS-homomorphism $g : S \to M$, where $K \leq S$. Unfortunately, our characterization is of general nature, that means, the result is not useful enough. The purpose of this section is to find a sufficient conditions under which we can easily say that an extension exists or not. For this reason we perform some speciali- zations (simple extensions, retractions) and a generalization (meet-semilattices). (See the discussion in the preceding section.)

Proposition 5.1. *Let $f : T \to M$ be a homomorphism of non-trivial bounded meet-semilattices. Assume that the bounded meet-semilattice $S = T[x]$ is a simple extension of T and u is an element of M. Moreover, assume that the element $\mathrm{Pr}_T^S(x)$ exists and, that we have a retraction $\alpha : T[x] \to T$, that means, $\alpha(a) = a$ for any $a \in T$, such that $\alpha(x) = \mathrm{Pr}_T^S(x)$. Then there exists a meet-semilattice homomorphism*

$$g : S = T[x] \to M$$

extending f and mapping x to $u \in M$ if and only if

$$a \leq x \text{ in } S \text{ and } a \in T \text{ imply } f(a) \leq u \leq f(\mathrm{Pr}_T^S(x)) \text{ in } M.$$

Proof. Necessity of the condition is obvious. As to sufficiency, it is known that an arbitrary element $v \in S$ can be written in the form

$$v = a \wedge x^r,$$

where $r \in \{0,1\}$ and $a \in T$. (Note that $x^0 = 1$ and $x^1 = x$.) Now, we can define

$$g : S \to M$$

by

$$g(v) = f(a) \wedge u^r.$$

First we have to show that g is well-defined, that is,

$$c \wedge x^r = d \wedge x^s \quad \text{implies} \quad f(c) \wedge u^r = f(d) \wedge u^s,$$

for $c, d \in T$. We have to verify two cases only:

$$c \wedge x = d \wedge x \quad \text{and} \quad c = d \wedge x.$$

Writing $\mathrm{Pr}(x)$ for $\mathrm{Pr}_T^S(x)$ we get in the first event

$$\alpha(c \wedge x) = c \wedge \mathrm{Pr}(x) = d \wedge \mathrm{Pr}(x) = \alpha(d \wedge x),$$

by the hypothesis on α. Therefore,

$$f(c) \wedge f(\mathrm{Pr}(x)) = f(c \wedge \mathrm{Pr}(x)) = f(d \wedge \mathrm{Pr}(x)) = f(d) \wedge f(\mathrm{Pr}(x)),$$

as f is a homomorphism. Since $u \leq f(\mathrm{Pr}(x))$, we obtain

$$f(c) \wedge u = f(d) \wedge u.$$

Considering the second case $c = d \wedge x$, we see that $c \leq x$. Hence $f(c) \leq u$, by the hypothesis on f. Using the same reasoning as above, we obtain

$$f(c) = f(d) \wedge u,$$

and g is well-defined. The element 0 in S can be expressed as $0 = 0 \wedge x$. Therefore,

$$g(0) = f(0) \wedge u = 0$$

in M. Similarly, $g(1) = 1$. Now, it can be readily shown that g is a meet-semilattice homomorphism extending f with the required properties.

Lemma 5.2. *Let $S = K[x]$ be a simple extension of PCS's. Assume that there exists $\mathrm{Pr}_K^S(x)$. Then there exists $\mathrm{Pr}_{S_1}^S(x)$ (for S_1 see Section 3) and*

$$\mathrm{Pr}_{S_1}^S(x) = x^{**} \wedge \mathrm{Pr}_K^S(x).$$

Proof. Clearly, $x \leq x^{**} \wedge \mathrm{Pr}_K^S(x) \in S_1$. On the other hand, let $x \leq v$ for some $v \in S_1$. By Theorem 3.2, $v = a \wedge t$ for some $a \in B(S)$ and $t \in K$. Now, $x \leq a \wedge t$ implies $x^{**} \leq a$ in $B(S)$ and $x \leq t$ in K. Hence

$$x^{**} \wedge \mathrm{Pr}_K^S(x) \leq a \wedge t = v.$$

As a consequence of these results we have

Theorem 5.3. *Let K, M and S be PCS's, let $S = K[x]$ be a simple extension of K for some $x \in S$ and let $u \in M$. Let $f_0 : K \to M$ be a PCS-homomorphism. Assume that the element $\mathrm{Pr}^S_K(x)$ exists and that we have (in the notation of Section 3) a retraction $\alpha : S_1[x] \to S_1$ such that $\alpha(x) = x^{**} \wedge \mathrm{Pr}^S_K(x)$. Then there exists a PCS-homomorphism*

$$g : S \to M$$

extending f_0 and mapping x to $u \in M$ if and only if

(i) *there exists a meet-semilattice homomorphism $f_1 : S_1 \to M$ which is an extension of $f_0 \cup h$ (see Theorem 4.1) and, we have*

$$f_1(x^{**}) = h(x^{**}) = u^{**},$$

(ii) *$t \leq x$ in S and $t \in S_1$ imply $f_1(t) \leq u \leq f_1(\mathrm{Pr}^S_{S_1}(x))$ in M.*

Proof. Suppose that $g : S \to M$ is an extension of f_0 such that $g(x) = u$. Since g is a PCS-homomorphism, condition (ii) follows easily. Condition (i) is a consequence of Theorem 4.1.

To prove the remaining half, let us suppose (i) and (ii). We shall proceed by Theorem 4.1. We start by establishing a Boolean homomorphism $h : B(S) \to B(M)$ which is an extension of $(f_0)_B$ (see Theorem 4.1) such that $h(x^{**}) = u^{**}$. It is easy to check that $[B(K) \cup \{x^{**}\}]_{Bool} = B(S)$. Moreover, from (ii) and the hypothesis that $\mathrm{Pr}^S_K(x)$ exists, it follows that

$$a^{**} \leq x^{**} \leq b^{**} \text{ in } S \text{ implies } f_0(a^{**}) \leq u^{**} \leq f_0(\mathrm{Pr}^S_K(x)^{**}) \leq f_0(b^{**}) \text{ in } M$$

for any $a, b \in K$. Now we can apply ([9], Corollary 5.8) of Sikorski's extension criterion for Boolean algebras. It does ensure that there is a Boolean homomorphism $h : B(S) \to B(M)$ extending $(f_0)_B : B(K) \to B(M)$ such that $h(x^{**}) = u^{**}$.

By (i) we see that $f_1 : S_1 \to M$ is a meet-semilattice homomorphism extending $f_0 \cup h$. It remains to show that there exists a meet-semilattice homomorphism $g : S \to M$ extending $f_1 \cup \{(x, u)\}$. Evidently, $S = S_1[x]$ is a simple meet-semilattice extension. Now, we can apply Proposition 5.1. By Lemma 5.2 and the hypothesis that f_1 ia a meet-semilattice homomorphism, we get

$$u^{**} \wedge f_0(\mathrm{Pr}^S_K(x)) = h(x^{**}) \wedge f_1(\mathrm{Pr}^S_K(x)) = f_1(\mathrm{Pr}^S_{S_1}(x)).$$

Now, setting T for S_1 in (ii), we get the main condition of Proposition 5.1. It follows that there exists a meet-semilattice homomorphism $g : S \to M$ extending $f_1 \cup \{(x, u)\}$. Ultimately Theorem 4.1 implies that g is a PCS-homomorphism, and the proof of the theorem is complete.

References

1. Balbes, R. and Dwinger, Ph., *Distributive lattices*, Univ. Missouri Press, Missouri, 1974.
2. Frink, O., *Pseudo-complements in semi-lattices*, Duke Math. J. *29*(1962), 505-514.

3. Grätzer, G., *General lattice theory*, Birkhäuser Verlag, Basel, Boston, Berlin, 1998 (sec. ed.).
4. Howie, J. M., *An introduction to semigroup theory*, Acad. Press, London, N.Y., San Francisco, 1976.
5. Jones, G. T., *Projective pseudocomplemented semilattices*, Pacific J. Math. *32* (1974), 443-456.
6. Jónsson, B., *Relatively free products of lattices*, Algebra Univ. *1* (1972), 362-371.
7. Katriňák, T., *Free p-algebras*, Algebra Univ. *15* (1982), 176-182.
8. Katriňák, T. and Guričan, J., *Projective extensions of bounded semilattices*, Algebra Univ. *59* (2008), 97-110.
9. Koppelberg, S., *Hanbook of Boolean algebras*, vol. *1* of: Monk, J. D. with R. Bonnet (eds.) North-Holland, Amsterdam etc. 1989.

Katedra AGDM FMFI, Comenius University, Mlynská dolina, 842 15 Bratislava, Slovakia

E-mail address, T. Katriňák: katrinak@fmph.uniba.sk

E-mail address, J. Guričan: gurican@fmph.uniba.sk