

Projective extensions of bounded semilattices

T. KATRÍŇÁK AND J. GURIČAN

Dedicated to George Grätzer and E. Tamás Schmidt on their 70th birthdays

ABSTRACT. Our aim is to characterize the projective extensions of bounded semilattices. As a consequence we get a new description of projective extensions of semilattices.

1. Introduction

In [8] we characterized projective extensions of semilattices. The case of bounded semilattices remained open. In this note we shall fill in the gap and describe the projective extensions of bounded semilattices. Moreover, as a consequence, a new description of projective extensions of semilattices is also obtained.

2. Preliminaries

We shall work with bounded semilattices. The basic terminology and notation will mostly agree with the one in [8] (see also [1]–[3] or [5] for further references). By a *bounded semilattice* we mean an algebra $S = (S; \cdot, 0, 1)$ of type $(2, 0, 0)$ such that $(S; \cdot)$ is a semilattice satisfying the following identities:

$$x1 = x \text{ and } x0 = 0.$$

(If no confusion is likely to arise, we will also write xy instead of $x \cdot y$.) It is well known that S has an ordering defined by $x \leq y$ if and only if $xy = x$. In this ordering, 0 is the smallest and 1 the largest element of S , respectively. A bounded semilattice is called *nontrivial*, if $0 \neq 1$. The class of all bounded semilattices is equational.

If S and T are bounded semilattices, the mapping $f: S \rightarrow T$ is called a $\{0,1\}$ -*homomorphism* whenever f is a semilattice homomorphism such that $f(0) = 0$ and $f(1) = 1$.

Presented by F. Wehrung.

Received January 3, 2007; accepted in final form April 25, 2007.

2000 *Mathematics Subject Classification*: Primary: 06B25 ; Secondary: 08B30.

Key words and phrases: Projective extension, coproduct, projective semilattice, semilattice with identity.

The research of both authors was supported by VEGA Grant 1/3020/06 of the Slovak Republic.

As has been shown in [9], the coproduct (free product) of algebras is crucial for investigating projective extensions of algebras. In this paper we shall examine coproducts of bounded semilattices (see also [1], [3], [5], or [8]).

We begin by recalling some definitions. Let S be a bounded semilattice and let $X \subseteq S$. We denote by $[X]$ the subalgebra of S generated by X . Note, that $[\emptyset] = \mathbf{2}$, where $\mathbf{2}$ means the two-element bounded semilattice. Let S_i , $i \in I$, be bounded semilattices. Then a bounded semilattice S together with a family of $\{0,1\}$ -homomorphisms $\epsilon_i: S_i \rightarrow S$, for $i \in I$, is a *coproduct* of the S_i , $i \in I$, if

- (i) $S = [\bigcup(\epsilon_i(S_i) \mid i \in I)]$,
- (ii) if T is any bounded semilattice and $\varphi_i: S_i \rightarrow T$ is a $\{0,1\}$ -homomorphism for every $i \in I$, then there exists a $\{0,1\}$ -homomorphism $\varphi: S \rightarrow T$ satisfying $\varphi_i = \varphi \epsilon_i (= \epsilon_i \circ \varphi)$ for every $i \in I$.

(Notice carefully that a composition $\alpha\beta$ of mappings means first β , than α .) It is well known, that by (i), the $\{0,1\}$ -homomorphism $\varphi: S \rightarrow T$ of (ii) is necessarily unique. Note that the coproducts of bounded semilattices exist and are unique up to isomorphism. In addition, a coproduct S of the S_i 's is called a *free product* of S_i , $i \in I$, if the $\{0,1\}$ -homomorphisms $\epsilon_i: S_i \rightarrow S$, $i \in I$, are *embeddings*, i.e., monomorphisms. It is easy to show that a free product of bounded semilattices exists if and only if the semilattices S_i 's are nontrivial.

We shall denote the coproduct of two bounded semilattices B and C by $B \oplus C$. Before we describe $B \oplus C$, we need some elementary ideas and definitions.

Lemma 2.1. *Let S be a nontrivial bounded semilattice. Let $\emptyset \neq J \subseteq S$ be a down-set of S , that is, $x \leq y$ in S and $y \in J$ imply $x \in J$. Then there exists a congruence relation θ on S defined as follows:*

$$x \equiv y(\theta) \text{ if and only if } x, y \in J \text{ or } x = y.$$

In addition, S/θ is a bounded semilattice, which is nontrivial, whenever $J \neq S$.

Proof. Straightforward. □

Let B and C be nontrivial bounded semilattices. Let $(B - \{0\}) \times (C - \{0\})$ denote the Cartesian product of the sets $B - \{0\}$ and $C - \{0\}$. Consider the set

$$B + C = ((B - \{0\}) \times (C - \{0\})) \cup \{(0, 0)\}.$$

A multiplication can be introduced on $B + C$ by the rule that

$$(a, m) \cdot (b, n) = \begin{cases} (0, 0) & \text{if } ab = 0 \text{ or } mn = 0, \\ (ab, mn) & \text{otherwise.} \end{cases}$$

Lemma 2.2. *$B + C$ with the given multiplication forms a bounded semilattice.*

Proof. Consider the direct product $B \times C$ of bounded semilattices B and C . Then the subset $(B \times \{0\}) \cup (\{0\} \times C)$ is a down-set of $B \times C$. By Lemma 2.1, there exists a congruence relation θ defined on $B \times C$. Clearly, $(B \times C)/\theta$ is a bounded semilattice. It is easy to verify that $\varphi: (B \times C)/\theta \rightarrow B + C$, where

$$\varphi: [(a, m)]\theta \mapsto \begin{cases} (0, 0) & \text{if } a = 0 \text{ or } m = 0, \\ (a, m) & \text{otherwise.} \end{cases}$$

is a bijection compatible with the multiplication on $B + C$. Thus, $B + C$ is a bounded semilattice and $B + C \cong (B \times C)/\theta$. $\square$

Proposition 2.3. *Let B and C be nontrivial bounded semilattices. Then $B + C$ is a free product of B and C , i.e., $B \oplus C \cong B + C$.*

Proof. It is easy to see that $\epsilon_B: B \rightarrow B + C$ defined by

$$\epsilon_B: a \mapsto \begin{cases} (0, 0) & \text{if } a = 0, \\ (a, 1) & \text{otherwise,} \end{cases}$$

is an embedding. The same is true for $\epsilon_C: C \rightarrow B + C$ defined by

$$\epsilon_C: m \mapsto \begin{cases} (0, 0) & \text{if } m = 0, \\ (1, m) & \text{otherwise.} \end{cases}$$

Since $(a, 1) \cdot (1, m) = (a, m)$ in $B + C$, we have $[\epsilon_B(B) \cup \epsilon_C(C)] = B + C$. Eventually, let T be an arbitrary bounded semilattice, and let $\varphi_B: B \rightarrow T$ and $\varphi_C: C \rightarrow T$ be $\{0, 1\}$ -homomorphisms. Now, we can define $\varphi: B + C \rightarrow T$ by the rule

$$\varphi: (a, m) \rightarrow \varphi_B(a) \cdot \varphi_C(m).$$

It is easy to check that φ is a $\{0, 1\}$ -homomorphism and $\varphi_B = \varphi \epsilon_B$ and $\varphi_C = \varphi \epsilon_C$. Thus, $B + C$ is a free product of B and C . $\square$

Definition 2.4. A nontrivial bounded semilattice S is called *dense* if it satisfies the property: $xy = 0$ implies $x = 0$ or $y = 0$.

The structure of $B + C$ simplifies considerably in the case of dense semilattices:

Corollary 2.5. *$B + C$ is a dense semilattice if and only if B and C are dense semilattices as well.*

In what follows we will assume that B and C are subalgebras of $B \oplus C$ and that ϵ_B and ϵ_C are inclusion mappings.

3. Projective extensions

Some preliminaries are necessary before we can describe projective extensions of bounded semilattices. If F is a (bounded) semilattice and $X \subseteq F$, then F is said to be a *free (bounded) semilattice freely generated by X* , if $[X] = F$ and any mapping $f: X \rightarrow S$, where S is a (bounded) semilattice, can be extended to a $(\{0,1\})$ -homomorphism $g: F \rightarrow S$. Notation: $\text{FS}(X)$ ($\text{FS}_{01}(X)$), or simply F .

We now immediately deduce: $\text{FS}_{01}(X)$ is obtained from $\text{FS}(X)$ by adjoining to it a zero element 0 and a unit one 1, even if such elements already exist in $\text{FS}(X)$. In addition, $\text{FS}_{01}(X)$ is a dense semilattice.

The next result describes free semilattices (see [4, Theorem 4.3]):

Proposition 3.1. *Let X be a nonempty set. Then $\text{FS}(X)$ is the set of all cofinite subsets Y of X such that $Y \neq X$, ordered by set inclusion.*

Let S and K be bounded semilattices and let K be a subalgebra of S , which means S is an extension of K , or equivalently, K can be embedded by inclusion mapping into S . (Notation: $K \leq S$.)

Suppose again $K \leq S$. Then S is called a *projective extension* of K , written $K \leq_{\text{proj}} S$, if there exist a free bounded semilattice $F = \text{FS}_{01}(Z)$ (freely generated by Z) and $\{0,1\}$ -homomorphisms

$$e: S \rightarrow K \oplus F, \quad q: K \oplus F \rightarrow S,$$

such that $qe = \text{id}(S)$ and the restrictions $e \upharpoonright K$ and $q \upharpoonright K$ are equal $\text{id}(K)$. See the following diagram:

$$\begin{array}{ccccc} K & \xrightarrow{\epsilon_K} & K \oplus F & \xleftarrow{\epsilon_F} & F \\ & & q \downarrow & \uparrow e & \\ & & S & & \end{array}$$

Recall that a bounded semilattice S is called *projective* if S is a retract of a free bounded semilattice F . (Notice that a projective bounded semilattice is dense.) We can express the same thing in other words: S is projective if and only if $\text{FS}_{01}(\emptyset) \leq_{\text{proj}} S$, or equivalently, $\mathbf{2} \leq_{\text{proj}} S$.

We remark finally that in [8] we succeeded in finding a construction of all dense semilattices S that are a projective extensions of a given dense semilattice K .

We now have the first observation:

Lemma 3.2. *Let $K \leq_{\text{proj}} S$ for bounded semilattices K and S . Assume $u \in S - K$. Then $c \leq u$ in S for $c \in K$ implies $c = 0$, i.e., $K - \{0\} = [K - \{0\}]$ is an upper subset of S .*

Proof. The statement follows from the fact that $K - \{0\}$ is an upper-subset of $K \oplus F$. $\square$

The following observations are either well known or easy to prove.

Lemma 3.3. *Let $K \leq_{\text{proj}} S$ for bounded semilattices. Then S is dense if and only if K is dense. Moreover, if K is dense, then $K - \{0\}$ is a filter of S .*

Lemma 3.4. *Let K and M be bounded semilattices. Assume that M is dense. Then there exists an epimorphism*

$$\pi_K: K \oplus M \rightarrow K$$

satisfying $\pi_K \upharpoonright K = \text{id}(K)$ and $\pi_K(c) = 1$, whenever $c \in M - \{0\}$.

For our final result we need the following concept: Let K and S be bounded semilattices such that $K \leq S$. Then K is said to be *relatively complete* in S , if for each $b \in S$ there exists a smallest $a \in K$ such that $b \leq a$. In notation:

$$a = \text{Pr}(b) = \text{Pr}_K^S(b) = \min\{x \in K \mid b \leq x\}.$$

Write $K \leq_{\text{rc}} S$ if K is relatively complete in S . (See also [7], or [9] for relatively complete Boolean algebras.)

Lemma 3.5. *Let K , M , and S be bounded semilattices. Assume that M is dense. Then*

- (i) $K \leq_{\text{rc}} K \oplus M$ and $\text{Pr}_K^{K \oplus M}: K \oplus M \rightarrow K$ is a $\{0, 1\}$ -homomorphism;
- (ii) If $K \leq_{\text{proj}} S$, then $K \leq_{\text{rc}} S$ and $\text{Pr}_K^S: S \rightarrow K$ is a $\{0, 1\}$ -homomorphism.

Proof. (i) By Lemma 3.4 and our convention about ϵ_K and ϵ_M , there exists a homomorphism $\pi_K: K \oplus M \rightarrow K$ such that $\pi_K \upharpoonright K = \text{id}(K)$ and $\pi_K(u) = 1$ for every $0 \neq u \in M$, and $\pi_K(0) = 0$. According to Proposition 2.3 every element $0 < u \in K \oplus M$ can be expressed as $u = (c, v) = (c, 1) \cdot (1, v) = c \cdot v$ for some $c \in K$ and $v \in M$. Therefore, $u \leq \pi_K(u) = c$ for $u \neq 0$. On the other hand, let $u \leq d$ for some $d \in K$. Hence, $u \leq \pi_K(u) \leq \pi_K(d) = d$. Thus $\pi_K = \text{Pr}_K^{K \oplus M}$, and the proof of (i) is complete. The part (ii) follows from (i), as S is a retract of $K \oplus F$ for some free bounded semilattice F . Clearly F is dense. All this is leading to

$$\text{Pr}_K^S = \text{Pr}_K^{K \oplus F} e,$$

where $e: S \rightarrow K \oplus F$. $\square$

4. Finite factorizations

We say that an element u of a bounded semilattice S is *irreducible* if, for all $x, y \in S$,

$$u = xy \text{ implies } u = x \text{ or } u = y.$$

Moreover, an element $p \in S$ is called (*weakly*) *prime* if,

$$xy \leq p \quad (0 < xy \leq p) \quad \text{implies} \quad x \leq p \quad \text{or} \quad y \leq p.$$

Proposition 4.1. *Let $K \leq_{\text{proj}} S$ for bounded semilattices.*

- (i) *Then $u \in S - K$ is an irreducible element of S if and only if there exists an irreducible element $0 < s \in \text{FS}_{01}(X)$ such that*

$$u = q(s) \quad \text{and} \quad e(u) \leq s,$$

where e and q denote the homomorphisms from the definition of projective extension.

- (ii) *$u \in S - K$ is irreducible in S if and only if u is weakly prime in S .*

Proof. (i) First recall that by our convention, K and $F = \text{FS}_{01}(X)$ are subalgebras of $K \oplus F$. Therefore, by Proposition 2.3, every element $w \in K \oplus F$ can be written as $w = \prod(y : y \in T)$, where $T \subseteq K \cup X$ is finite. Assume that $u \in S - K$ is irreducible. Then there exists a finite $T \subseteq K \cup X$, such that $e(u) = \prod(y : y \in T)$, $e(u) \notin K$, and consequently, $T \cap X \neq \emptyset$. Therefore, $u = (qe)(u) = \prod(q(y) : y \in T)$, and $u = q(s)$ for some $s \in (T \cap X)$, as u is irreducible. Since $s \in (T \cap X)$, we deduce $e(u) \leq s = (1, s)$. It is easy to verify that $X \cup \{0\}$ is the set of all irreducible elements of F (see Proposition 3.1).

Conversely, suppose that $u = q(s)$ and $e(u) \leq s$ for some $s \in X$ and $u \in S - K$. By Proposition 2.3 we get $(0, 0) < e(u) = (c, t) \leq (1, s) = s$ in $K \oplus F$. Assume that $u = vw$ in S . Then $(0, 0) < e(u) = e(v) \cdot e(w) \leq (1, s)$ in $K \oplus F$. Using Proposition 2.3, we have $e(v) = (a, v_1)$ and $e(w) = (b, w_1)$. Hence, $(0, 0) < e(u) = (ab, v_1w_1) \leq (1, s)$ in $K \oplus F$ and $v_1w_1 \leq s$ in F . But $s \in X$ is prime in F by [4, Theorem 4.3]. This yields $v_1 \leq s$ or $w_1 \leq s$ in F . Assume first $v_1 \leq s$. Then $e(v) = (a, v_1) \leq (1, s) = s$ in $K \oplus F$. Similarly we can establish $e(w) \leq (1, s)$ in the second case. Now, $v = (qe)(v) \leq q(s) = u$, which implies $v = u$ in the first event, and $w = u$ in the second one. Thus, u is irreducible.

(ii) Consider $0 < xy \leq u$ for $x, y \in S$ and $u \in S - K$. Assume that u is irreducible in S . Similarly as above we obtain $0 < e(x) \cdot e(y) \leq e(u)$ in $K \oplus F$. By virtue of Proposition 2.3, we can write $e(u) = (a, u_1)$, $e(x) = (b, x_1)$, $e(y) = (c, y_1)$ in $K \oplus F$. Rewriting our condition, we get

$$(0, 0) < (bc, x_1y_1) \leq (a, u_1) \leq (1, s)$$

for some $s \in X$, as u is irreducible (see (i)). Therefore, $x_1y_1 \leq s$ in F , which is a free bounded semilattice with the free generators X . From Proposition 3.1 we deduce that s is prime in F , hence $x_1 \leq s$ or $y_1 \leq s$. Assume that $x_1 \leq s$. Therefore, $e(x) \leq (1, s) = s$ in $K \oplus F$, and consequently $(qe)(x) = x \leq q(s) = u$, by (ii). Similarly in the second case, we get $y \leq u$. $\square$

Let $\text{MI}(S)$ denote the set of all irreducible elements of S . Of more importance is the following set $\text{MI}([u])$, which means $[u] \cap \text{MI}(S) \cap (S - K)$, or in other words, the set of all irreducibles $v \in S - K$ such that $u \leq v$.

Lemma 4.2. *Let $K \leq_{\text{proj}} S$ for bounded semilattices. Let $u \in S - K$ be irreducible. Then the set $\text{MI}([u])$ is finite.*

Proof. Consider $u, v \in S - K$ such that both elements are irreducible in S . Suppose that $u \leq v$ in S . By Proposition 4.1 there exist irreducibles $s, t \in F$, i.e., $s, t \in X$, such that

$$e(u) \leq s = (1, s) \quad \text{and} \quad e(v) \leq t = (1, t)$$

in $K \oplus F$, and in addition, $q(s) = u$ and $q(t) = v$. Since $e(u) \leq e(v)$, we have $e(u) \leq s, t$. As $e(u) = (a, u_1)$ in $K \oplus F$, we deduce that $u_1 \leq s, t$ in F , which is free. But $[u_1] \cap X$ is a finite subset of F (see Proposition 3.1). Thus, the set $\text{MI}([u])$ is also finite, and the proof is complete. $\square$

If u and v are elements of a bounded semilattice S , then we say that v is a *factor* of u , if there exists an element $w \in S$ such that $u = vw$. (Notation: $v \mid u$.) It is easy to see that $v \mid u$ is equivalent to $u \leq v$. If, in addition, K is a subalgebra of S , then elements $u, v \in S$ are called *K -associated*, if $0 < c \cdot u = c \cdot v$ for some $0 < c \in K$. In this case we write $u \approx_K v$ (see [8]).

Definition 4.3. Let $K \leq S$ be bounded semilattices and $u, v \in S$. If $v \mid u$ and v is neither an element from K nor a K -associate of u , then we say that v is a *proper factor* of u .

The next result requires only routine verification, and the proof is omitted.

Lemma 4.4. *Let $K \leq_{\text{proj}} S$ for bounded semilattices, $u, v \in S$ and $0 < u$. Let e and q denote the homomorphisms from the definition of projective extensions. Assume that $e(u) = (a, u_1)$ and $e(v) = (b, v_1)$ in $K \oplus F$. Then $v \mid u$ in S if and only if $b \mid a$ in K and $v_1 \mid u_1$, i.e., $u_1 \leq v_1$, in F . In addition,*

- (i) $v \mid u$ and $v \in K$ if and only if $b \mid a$ and $v_1 = 1$;
- (ii) $v \mid u$ and $u \approx_K v$ if and only if $b \mid a$ and $u_1 = v_1$;
- (iii) v is a proper factor of u if and only if $b \mid a$ and $u_1 < v_1 < 1$.

Definition 4.5. Let $K \leq S$ be bounded semilattices. We say that S satisfies the *(S, K) -ascending chain condition*, if S contains no infinite sequences $w_1, w_2, \dots$ with the property that each w_{i+1} is a proper factor of w_i .

The crucial property of projective extensions is given by the following lemma.

Lemma 4.6. *Let $K \leq_{\text{proj}} S$ for bounded semilattices. Then S satisfies the (S, K) -ascending chain condition.*

Proof. Suppose to the contrary that there exists an infinite sequence $w_1, w_2, \dots$ in S having the property that each w_{i+1} is a proper factor of w_i . Since $K \leq_{\text{proj}} S$, there exist a free bounded semilattice $F = \text{FS}_{01}(X)$, and $\{0,1\}$ -homomorphisms $e: S \rightarrow K \oplus F$ and $q: K \oplus F \rightarrow S$ satisfying conditions required for a projective extension. Now, we obtain a new infinite sequence $e(w_1), e(w_2), \dots$ in $K \oplus F$. It can easily be verified that each $e(w_{i+1})$ is a proper factor of $e(w_i)$. (Note that $K \leq K \oplus F$.) Applying Proposition 3.1 we get $e(w_1) = (c_1, v_1)$, $e(w_2) = (c_2, v_2)$, $\dots$ in $K \oplus F$. This is equivalent to the fact that each (c_{i+1}, v_{i+1}) is a proper factor of (c_i, v_i) . It follows from Lemma 4.4 that $0 < v_1 < v_2 < \dots$ in F , or equivalently, $v_1, v_2, \dots$ is an infinite properly ascending chain in F . But this is impossible by Proposition 3.1. $\square$

Lemma 4.7. *Let $K \leq_{\text{proj}} S$. Let $u \in S - K$ have no proper factors in S . Then there exist an irreducible element $v \in S$ and $a \in K$ such that $u = av$.*

Proof. Consider $u \in S - K$ having no proper factors. By the hypothesis there exist a free bounded semilattice $F = \text{FS}_{0,1}(X)$, $\{0,1\}$ -homomorphisms $e: S \rightarrow K \oplus F$, and $q: K \oplus F \rightarrow S$ satisfying the conditions required by projective extensions.

Now, $e(u) = (a, u_1) = \prod (y : y \in T)$, where $T \subseteq K \cup X$ is finite. Since $u \in S - K$, we have $e(u) \in (K \oplus F) - K$. Hence, $T \cap X \neq \emptyset$. The above representation of $e(u)$ implies that

$$u = (qe)(u) = \prod (q(y) : y \in T)$$

in S . Evidently, $y = q(y) \in K$ for every $y \in K$. This implies that there exists $s \in T \cap X$ such that $v = q(s) \in S - K$ and $e(u) \leq s = (1, s)$ in $K \oplus F$. Clearly v is a factor of u . By our hypothesis on u , we get $u \approx_K v$. Now, it is easy to show from Lemma 4.4 that $e(u) = a \cdot e(v)$, and consequently, $u = av$ in S .

It remains to show that v is irreducible. We can apply Proposition 4.1. Above we have shown that $v = q(s)$ and $e(u) \leq s$ for some $s \in X$. Using Proposition 3.1 we can write $(0, 0) < e(v) = (b, v_1)$ in $K \oplus F$. Therefore, $e(u) = a \cdot e(v) = (ab, v_1) = (a, v_1) \leq (1, s)$ in $K \oplus F$. Hence, $v_1 \leq s$ in F , and consequently, $e(v) = (b, v_1) \leq (1, v_1) \leq (1, s)$ in $K \oplus F$. Thus, $e(v) \leq s = (1, s)$ and the proof is complete. $\square$

Proposition 4.8. *Let $K \leq_{\text{proj}} S$ for bounded semilattices. Then every element $u \in S - K$ has a finite factorization*

$$u = cp_1 \cdots p_k,$$

where $c \in K$ and $p_1, \dots, p_k \in S - K$ are irreducible elements of S .

Proof. Let $u \in S - K$ be given. By hypothesis there exist a free bounded semilattice $F = \text{FS}_{0,1}(X)$ and, $\{0,1\}$ -homomorphisms $e: S \rightarrow K \oplus F$ and $q: K \oplus F \rightarrow S$ satisfying the conditions required by projective extensions.

Similarly as in the proof of Lemma 4.7, $e(u) = (a, u_1) = \prod(y : y \in T)$, where $T \subseteq K \cup X$ is finite. Since $u \in S - K$, we have $e(u) \in (K \oplus F) - K$. Hence, $T \cap X \neq \emptyset$. From the representation of $e(u)$ we deduce that

$$u = (qe)(u) = \prod(q(y) : y \in T)$$

in S . Let T_1 be the set of all $y \in T$ such that $q(y) \in K$. Since K is a subalgebra of S , we get $c = \prod(q(y) : y \in T_1) \in K$ and $c \mid a$. Moreover, let $T_2 = T - T_1$. Trivially, $T_2 \subseteq T \cap X$. Now, we have

$$u = c \cdot \prod(q(y) : y \in T_2).$$

Notice that $q(y) \in S - K$ for $y \in T_2$.

Clearly, the set $M = \{q(y) : y \in T_2\}$ is a finite subset of S . We can consider the subset $M_{\min}$ of all minimal elements from M . An alternative description is given by $M_{\min} = \{q(y) : y \in T_{\min}\}$ for some $T_{\min} \subseteq T_2$. It is easy to verify that

$$u = c \cdot \prod(q(y) : y \in T_{\min}).$$

We shall show that this is the required factorization of u . By Proposition 4.1, we have only to prove that each $q(y) \in M_{\min}$ is irreducible in S . Really, take $s \in T_{\min}$. Since $s \in X$, we see that s is a prime element of F (see Proposition 3.1 and [4, Theorem 4.3]). In addition,

$$e(u) = c \cdot \prod(e(q(y)) : y \in T_{\min}) = \prod(y : y \in T) \leq s = (1, s)$$

in $K \oplus F$. By this and the argument in the proof of Proposition 4.1, we have

$$e(q(y)) \leq s$$

for some $y \in T_{\min}$. Note, that $c \leq s$ is impossible, as K is an upper subset and $s \in X \subseteq F$. Hence,

$$q(y) = q(e(q(y))) \leq q(s)$$

in S . But $q(s)$ is a minimal element of M , i.e., $q(s) \in M_{\min}$. Therefore, $q(y) = q(s)$ and we have $e(q(s)) \leq s$ in $K \oplus F$. By Proposition 4.1 each $q(y)$ for $y \in T_{\min}$ is an irreducible element of S , and the required result follows. $\square$

5. The main result

Proposition 4.8 provides a method for factorization of an element. More precisely, let $K \leq S$ for bounded semilattices. We say that $u \in S$ has a *finite factorization* (into irreducibles), if there exist $c \in K$ and a finite set $I \subseteq M$, where $M = \text{MI}(S) \cap (S - K)$ such that $u = c \cdot \prod(x : x \in I)$. This factorization is *irredundant* if $u \neq c \cdot \prod(x : x \in I_1)$ for each $I_1 \subseteq I$ and $I_1 \neq I$. (If an element has a finite

factorization, then it has an irredundant factorization; by deleting superfluous irreducible elements, the original finite factorization can be refined to an irredundant one.)

About uniqueness of a factorization we have the following simple results:

Lemma 5.1. *Let $K \leq S$ be bounded semilattices. Let S satisfy the condition:*

(i) $K \leq_{rc} S$ and $\text{Pr}_K^S: S \rightarrow K$ is a $\{0, 1\}$ -homomorphism.

If $u = c \cdot \prod(x : x \in I)$ is a finite factorization of $u \in S$, then $\text{Pr}_K^S(u) \prod(x : x \in I)$ is also a finite factorization of u .

Lemma 5.2. *Let $K \leq S$ for bounded semilattices. Let S satisfy the condition (i) from Lemma 5.1 and the following ones:*

- (ii) $K - \{0\}$ is an upper subset of S ,
- (iii) $[M \cup K] = S$,
- (iv) every $u \in M$ is weakly prime in S .

Then

- (1) $u = 0$ if and only if $\text{Pr}_K^S(u) = 0$.
- (2) If $u = \text{Pr}_K^S(u) \prod(x : x \in I)$ and $v = \text{Pr}_K^S(v) \prod(y : y \in J)$ are finite irredundant factorizations, then $u \leq v$ in S if and only if $\text{Pr}_K^S(u) \leq \text{Pr}_K^S(v)$ in K and for every $y \in J$ there exists $x \in I$ such that $x \leq y$.
- (3) If $u, v \in S$ are expressed as in (2), $u \neq 0$ and $v \neq 0$, then $u = v$ if and only if $\text{Pr}_K^S(u) = \text{Pr}_K^S(v)$ and $I = J$.

It is convenient now to state the theorem towards which we are working.

Theorem 5.3. *Let $K \leq S$ for bounded semilattices. Then $K \leq_{\text{proj}} S$ if and only if S satisfies the conditions (i)–(iv) from Lemmas 5.1 and 5.2, as well as the condition*

(v) $\text{MI}([u])$ is a finite set, whenever u is an irreducible element from $S - K$.

Proof. The necessity follows from Lemmas 3.2, 3.5, 4.2 and Propositions 4.1 and 4.8.

To prove the converse, let us consider the bounded semilattices $K \leq S$ satisfying the conditions (i) - (v). The case $K = S$ is trivial, so we can assume $K \neq S$. Denote by M the set $\text{MI}(S) \cap (S - K)$. Evidently, M is a partially ordered set of irreducibles of S lying in $S - K$. Take a set X of the same cardinality as the set M . There exists a bijective mapping $f: X \rightarrow M$, i.e., $f(x)$ is an irreducible element from $S - K$. Now, we can construct $F = \text{FS}_{0,1}(X)$ and a $\{0, 1\}$ -homomorphism $\varphi_F: F \rightarrow S$ extending f . In addition, set $\varphi_K: K \rightarrow S$ by the rule $\varphi_K = \text{id}(K)$. Eventually, we have a $\{0, 1\}$ -homomorphism $q: K \oplus F \rightarrow S$, extending φ_K and φ_F , and which is an epimorphism by (iii).

We have to show that there exists a $\{0, 1\}$ -homomorphism $e: S \rightarrow K \oplus F$ such that $qe = \text{id}(S)$ and $e \upharpoonright K = \text{id}(K)$. Since f is a bijection, there is $f^{-1}: M \rightarrow X$,

and consequently, we can consider the mapping $f^{-1}: M \rightarrow K \oplus F$, as $F \subseteq K \oplus F$. We now define a new mapping $g_0: M \rightarrow K \oplus F$ by the rule

$$g_0(u) = \text{Pr}_K^S(u) \prod (f^{-1}(v) : v \in \text{MI}([u])).$$

This is possible by (i) and (v). It is easy to verify that g_0 is isotone, which means $u \leq v$ implies $g_0(u) \leq g_0(v)$. In addition, $qg_0 = \text{id}(M)$. In the next step we extend g_0 to $g_1: K \cup M \rightarrow K \oplus F$ by the rule

$$g_1: x \mapsto \begin{cases} x & \text{if } x \in K, \\ g_0(x) & \text{otherwise.} \end{cases}$$

Since $K \cap M = \emptyset$, the definition of g_1 is correct. We claim that g_1 is again isotone. Indeed, take $u \leq v$ in $K \cup M$. We have only to verify two cases: $u \in K$ and $v \in M$, or $v \in K$ and $u \in M$. The first is impossible by (ii). In the second case we see that

$$u \leq \text{Pr}_K^S(u) \leq v$$

by (i). Therefore,

$$g_1(u) = g_0(u) \leq \text{Pr}_K^S(u) \leq \text{Pr}_K^S(v) = v = g_1(v)$$

in $K \oplus F$. Again, it is routine to show that $qg_1 = \text{id}(K \cup M)$.

Eventually, we extend g_1 to a $\{0, 1\}$ -homomorphism $e: S \rightarrow K \oplus F$. By virtue of (iii) we know that every element $u \in S$ has a finite factorization. Therefore, by Lemma 5.1, every element $u \in S$ has a finite irredundant factorization

$$u = \text{Pr}_K^S(u) \prod (x : x \in I),$$

for some finite $I \subseteq M$. We deduce from Lemma 5.2 that this factorization is unique. Now, we can define

$$e: u \mapsto g_1(\text{Pr}_K^S(u)) \prod (g_1(x) : x \in I) = \text{Pr}_K^S(u) \prod (g_1(x) : x \in I).$$

We claim that e is a $\{0, 1\}$ -homomorphism. Clearly, $e(0) = 0$ and $e(1) = 1$. Consider a second element $v \in S$ having the following irredundant factorization: $v = \text{Pr}_K^S(v) \prod (y : y \in J)$, where $J \subseteq M$ is finite. Assume first $uv = 0$ in S . Lemma 5.1 yields

$$\text{Pr}_K^S(uv) = \text{Pr}_K^S(0) = 0 = \text{Pr}_K^S(u) \cdot \text{Pr}_K^S(v).$$

Therefore, $e(uv) = e(0) = e(u) \cdot e(v)$. Suppose that $uv \neq 0$ in S . Hence,

$$e(uv) = \text{Pr}_K^S(uv) \prod (g_1(x) : x \in L) = \text{Pr}_K^S(uv) \prod (g_1(x) : x \in I \cup J),$$

where $L = \{x \in I \cup J : x \text{ is a minimal element of } I \cup J\}$. Now, it is easy to verify that $e(uv) = e(u) \cdot e(v)$. The restriction $e \upharpoonright K = \text{id}(K)$ and $qe = \text{id}(S)$ is routine, and the proof is complete. $\square$

Remark. Theorem 5.3 simplifies in the case of dense semilattices. It is now not hard to deduce that condition (ii) can be replaced in this case by

(ii') $K - \{0\}$ is a filter of S .

An alternative characterization of $K \leq_{\text{proj}} S$ for dense semilattices S is given in [8, Theorem 3.8].

Definition 5.4. Let $\gamma: S \rightarrow S$ be a $\{0, 1\}$ -endomorphism of a bounded semilattice S . Then we shall say that γ is a *closure morphism* of S , if it satisfies

- (i) $u \leq \gamma(u)$ for every $u \in S$ and
- (ii) $\gamma(u) = \gamma(\gamma(u)) = \gamma^2(u)$ for every $u \in S$.

A more striking characterization of $K \leq_{\text{proj}} S$ for dense semilattices is the following

Theorem 5.5. *Let $K \leq S$ for dense semilattices. Then $K \leq_{\text{proj}} S$ if and only if the following conditions are satisfied:*

- (a) $K \leq_{rc} S$ and $\text{Pr}_K^S: S \rightarrow K$ is a $\{0, 1\}$ -homomorphism,
- (b) $K - \{0\}$ is a filter of S ,
- (c) there exists a closure morphism $\gamma: S \rightarrow S$ such that $\gamma(S)$ is a projective bounded semilattice and $\gamma(S) \cap K = \{0, 1\}$,
- (d) $[\text{MI}(\gamma(S)) \cup K] = S$.

Proof. Suppose that $K \leq_{\text{proj}} S$. Part (a) follows from Theorem 5.3. For part (b) see Lemma 3.3. To prove (c) we have to define a closure morphism $\gamma: S \rightarrow S$. First notice that there exists a $\{0, 1\}$ -homomorphism

$$\pi_F: K \oplus F \rightarrow F$$

defined by $\pi_F \upharpoonright F = \text{id}(F)$ and $\pi_F(c) = 1$, whenever $c \in K - \{0\}$ and $\pi_F(0) = 0$ (see [8, Lemma 3.4]). Now, we can define $\gamma = q\epsilon_F \pi_F e: S \rightarrow S$ (see again [8]). In the proof of [8, Lemma 3.7(ii)] we have shown that γ is a closure morphism. Moreover, $\gamma q\epsilon_F: F \rightarrow \gamma(S)$ is a $\{0, 1\}$ -homomorphism. Consider the restriction $e_0 = e \upharpoonright \gamma(S)$. Obviously, $e_0: \gamma(S) \rightarrow K \oplus F$ is a $\{0, 1\}$ -homomorphism. Similarly, $\pi_F e_0: \gamma(S) \rightarrow F$ is a $\{0, 1\}$ -homomorphism. Eventually, we get

$$(\gamma q\epsilon_F)(\pi_F e_0) = \text{id}(\gamma(S)),$$

as $\gamma = \gamma^2$. Thus, $\gamma(S)$ is projective and $\gamma(u) = 1$ for $0 < u \in K$. To prove part (d) of the theorem, we recall that irreducible elements of a projective semilattice generate this semilattice (see [4, Theorem 5.3]). Hence, $\gamma(S) = [\text{MI}(\gamma(S))]$. Again, by [8, Lemma 3.4] we know that every $u \in S$ can be written as $u = c \cdot \gamma(u)$ for some $c \in K$, and (d) is established.

Conversely, suppose that (a)–(d) are satisfied. Since the case $K = S$ is trivial, we may assume that $K \neq S$. We shall show that conditions (i)–(v) of Theorem 5.3 are satisfied. Clearly, (i) and (ii) follow from (a) and (b). To prove (iii) and (iv), notice first that every weakly prime element of S is prime, as S is dense. We start with claim that $p \in \text{MI}(\gamma(S))$ is prime in S . Really, the case $p = 0$ is trivial, because S is dense. It remains to assume that $0 < p \in \text{MI}(\gamma(S))$. Consider $uv \leq p$ in S . Therefore,

$$\gamma(uv) = \gamma(u) \cdot \gamma(v) \leq \gamma(p) = p,$$

since γ is a closure morphism and $p \in \gamma(S)$. By virtue of [4, Theorem 5.3] and (c), p is a prime element of $\gamma(S)$. Hence $u \leq \gamma(u) \leq p$ or $v \leq \gamma(v) \leq p$ in S , as claimed. Notice, that we have shown more, namely that

$$\text{MI}(\gamma(S)) - \{0, 1\} \subseteq M = \text{MI}(S) \cap (S - K).$$

It follows by (d) that each $u \in S$ has a finite (irredundant) factorization into irreducibles. More precisely,

$$u = \text{Pr}_K^S(u) \prod (x : x \in I),$$

where $I \subseteq \text{MI}(\gamma(S))$ is finite. Since $x \in \text{MI}(\gamma(S))$ are also prime in S , we get by Lemma 5.2 that the given factorization is unique. Suppose now, that $u \in M$. By the above finite factorization of u we get $u = \text{Pr}_K^S(u)$ or $u = x$ for some $x \in I$ and $I \subseteq \text{MI}(\gamma(S))$, as u is irreducible in S . But $u = \text{Pr}_K^S(u)$ is impossible. Hence, $u = x$, and $u \in \text{MI}(\gamma(S))$. Eventually, we have proven

$$M = \text{MI}(\gamma(S)) - \{0, 1\} \quad \text{and} \quad M \cup K = \text{MI}(\gamma(S)) \cup K.$$

Consequently, (c) and (d) imply (iii) and (iv) from Theorem 5.3. It remains to verify (v). But this follows from [4, Theorem 5.3], as $\gamma(S)$ is a projective bounded semilattice (see (c)). Thus, we have $K \leq_{\text{proj}} S$ by Theorem 5.3, and the proof is complete. $\square$

Acknowledgement. We thank the referee for the careful reading of the paper and for the suggestion which led us to improve the proof of Proposition 4.8.

REFERENCES

- [1] R. Balbes and Ph. Dwinger, *Distributive Lattices*, Univ. Missouri Press, Columbia, 1974.
- [2] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups* Vol. **2**, Amer. Math. Soc., Providence, Rhode Island, 1967 (Russian Translation 1972).
- [3] G. Grätzer, *General Lattice Theory*, Birkhäuser Verlag, Basel–Stuttgart, 1978.
- [4] A. Horn and N. Kimura, *The Category of Semilattices*, *Algebra Universalis* **1** (1971), 26–38.
- [5] J. M. Howie, *An Introduction to Semigroup Theory*, L. M. S. Monographs, Vol. **7**, Academic Press, London, N.Y. and San Francisco, 1976.
- [6] N. Jacobson, *Lectures in Abstract Algebra* Vol. **1**, D. Van Nostrand Comp., Princeton, New Jersey, Toronto, New York, London, 1964.
- [7] B. Jónsson, *Relatively Free Products of Lattices*, *Algebra Universalis* **1** (1972), 362–371.

- [8] T. Katriňák, and J. Guričan, *Projective extensions of semilattices*, Algebra Universalis **55** (2006), 45–55.
- [9] S. Koppelberg, *Projective Boolean Algebras*, Chapter **20**, 741–774, in Vol. **3** of *Handbook of Boolean Algebra*, J. D. Monk and R. Bonnet, eds., North-Holland, Amsterdam, 1989.

T. KATRIŇÁK AND J. GURIČAN

Katedra AGDM, FMFI, Comenius University, Mlynská dolina, 84215 Bratislava, Slovakia

e-mail: katrinak@fmph.uniba.sk

e-mail: gurican@fmph.uniba.sk