Finite pseudocomplemented lattices: The spectra and the Glivenko congruence

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Abstract. Recently, Grätzer, Gunderson and Quackenbush have characterized the spectra of finite pseudocomplemented lattices solving a problem raised by G. Grätzer in his first monograph on lattice theory from 1971. In this note we discuss the tight connection between the spectra and the Glivenko congruence of finite pseudocomplemented lattices.

1. Introduction

The classes of pseudocomplemented semilattices and Boolean lattices are closely related. More precisely, with each pseudocomplemented semilattice P we associate the Boolean lattice

$$\mathrm{S}(P) = \mathrm{S}(1) = \{x^{**} : x \in P\}.$$

In addition, it is well-known that for every element a of P, the principal ideal $(a] = \{x \in P : x \leq a\}$ is again a pseudocomplemented semilattice. The corresponding Boolean lattice will be denoted by $\mathrm{S}(a)$. (It will soon follow from Lemma 3.4 and Theorem 3.5 that $|\mathrm{S}(a)| \leq |\mathrm{S}(P)|$, for all $a \in P$.) In summary, to every pseudocomplemented semilattice P one can assign the following family

$$\{\mathrm{S}(a) : a \in P\}$$

of Boolean lattices. A problem raised by G. Grätzer [4, Problem 22, p. 67] (and specified for pseudocomplemented semilattices) asked for a characterization of the family $\{\mathrm{S}(a) : a \in P\}$ of Boolean lattices.

There is a simplification of this problem for the class of finite pseudocomplemented lattices L, suggested in [6]: Let $\mathrm{S}(L)$ have exactly n atoms. We shall say that the *spectral length* of L is n, or we write $\mathrm{speclength}(L) = n$. Let B_i denote the finite Boolean lattice with i atoms. For a finite pseudocomplemented lattice L with spectral length n, define the *spectrum* of L, denoted by $\mathrm{spec}(L)$, as the sequence $\mathbf{s}=(s_0, \ldots, s_n)$, where

$$s_i = |\{a \in L : \mathrm{S}(a) \cong B_i\}|.$$

Observe that $s_0 = 1$.

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The preceding observations suggest the following new formulation of the question from [4]:

Problem. *Characterize the spectra of finite pseudocomplemented lattices.*

This has been solved in [6] as follows:

Theorem 1.1. *A sequence* $\boldsymbol{s}$=$(1, s_1, \dots, s_n)$ *of positive integers is the spectrum of a finite pseudocomplemented lattice iff the inequality*

$$\binom{n}{i} \leq s_i$$

holds for all $1 \leq i \leq n$.

The aim of this note is to discuss the close connection between the spectra and the Glivenko congruences on finite pseudocomplemented semilattices. In the last section we shall characterize the spectra of finite Stone lattices.

2. Preliminaries

In general we shall follow the notation and terminology of the monograph [5], but for the sake of clarity we collect in this section the definitions and known results which will be used throughout the paper.

A bounded meet-semilattice $P = (P; \wedge, 0, 1)$ is called a *pseudocomplemented semilattice* (= PCS) if each element $a \in P$ has a so-called *pseudocomplement* a^* with the property that $x \leq a^*$ iff $x \wedge a = 0$. If a PCS P is a lattice, then we shall call it a *pseudocomplemented lattice* (= PCL). Of course, we shall distinguish both concepts from the corresponding concepts of a *p-semilattice* and of a *p-lattice* (or *p-algebra*), respectively. More precisely, an algebra $(P; \wedge, ^*, 0, 1)$ is called a *p-semilattice,* if $(P; \wedge, 0, 1)$ is a bounded meet-semilattice and the unary operation * is the operation of pseudocomplementation. In addition, if $(L; \wedge, \vee, ^*, 0, 1)$ is an algebra such that $(L; \wedge, ^*, 0, 1)$ is a p-semilattice and $(L; \wedge, \vee, 0, 1)$ is a bounded lattice, then the algebra L is said to be a *p-lattice* (or *p-algebra*). A PCS P is said to be *non-trivial*, whenever $|P| \geq 2$. An element $a \in P$ is called *closed*, if $a = a^{**}$. Let $\mathrm{S}(P)$ denote the set of all closed elements of P, or in other words, the *skeleton* of P. The famous theorem of Glivenko-Frink [2], [3] (see also [1] or [4] and [5, Theorem I.6.4]) states

Theorem 2.1. *Let P be a p-semilattice. Then the algebra*

$$\mathrm{S}(P) = (\mathrm{S}(P); \sqcup, \wedge, ^*, 0, 1)$$

forms a Boolean algebra with respect to the join operation

$$a \sqcup b = (a^* \wedge b^*)^*.$$

Remark. Note, that the partial ordering of P partially orders $\mathrm{S}(P)$ and makes $\mathrm{S}(P)$ into a Boolean lattice. Therefore the operations $\wedge$, *, 0 and 1 in $\mathrm{S}(P)$, are the same as the original operations in P. However, if P happens to be a PCL, the join in P restricted to $\mathrm{S}(P)$ need not be the same as the join in $\mathrm{S}(P)$.

Clearly, a p-semilattice P is a Boolean algebra iff P satisfies the identity $x = x^{**}$.

Here are some rules of computation with * and $\wedge$ which we will use frequently (see [4]):

(1) $x \wedge x^* = 0$.
(2) $x \leq y$ implies that $x^* \geq y^*$.
(3) $x \leq x^{**}$.
(4) $x^* = x^{***}$.
(5) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$.
(6) $0^* = 1$ and $1^* = 0$.

Moreover, if P is actually a PCL, then

(7) $(x \vee y)^* = x^* \wedge y^*$,
(8) $x^* \vee y^* \leq (x \wedge y)^*$.

Observe that $\mathrm{S}(P) = \{x^* : x \in P\}$, which follows from (4).

There is an important supplement to the Glivenko-Frink theorem (see O. Frink [2, Theorem 2] or [1]):

Theorem 2.2. *The mapping $x \to x^{**}$ from a p-semilattice P into the Boolean algebra $\mathrm{S}(P)$ is a p-semilattice homomorphism of P onto $\mathrm{S}(P)$, which means that it is a mapping preserving meets, pseudocomplements and the 0 element. Moreover, it preserves joins when they exist.*

The mapping $x \to x^{**}$ on a p-semilattice P gives rise to the following binary relation $\Gamma(P)$ (or shortly Γ) defined on P :

$$x \equiv y \ (\Gamma) \ \text{ iff } \ x^{**} = y^{**} \ \text{ iff } \ x^* = y^*.$$

It is routine to verify that Γ is a p-semilattice congruence relation on P (or a p-lattice congruence relation, whenever P is a p-lattice) and we shall call it the *Glivenko congruence* on P. Observe that $[0]\Gamma = \{0\}$. Another important subset of P is the congruence class $[1]\Gamma$. Clearly,

$$[1]\Gamma = \{x \in P : x^* = 0\}$$

is a filter of P and the elements of $[1]\Gamma$ are called *dense*. The set $[1]\Gamma$ is also known as *filter $D(P)$ of dense elements.*

It is easy to verify the following alternative formulation of the Glivenko-Frink theorem:

Corollary 2.3. *Let P be a p-semilattice. Then*

$$P/\Gamma \cong \mathrm{S}(P).$$

3. More about Glivenko congruences

Lemma 3.1. *Let P be a* PCS *and let $a \in P$. Then $(a] = [0, a]$ is again a pseudocomplemented semilattice and $x^+ = x^* \wedge a$ is the pseudocomplement for any x in $(a]$. In addition, $x^{++} = x^{**} \wedge a$.*

Proof. The proof is obvious. □

Now, define $\mathrm{S}(a) = \mathrm{S}((a])$ and $\Gamma_a = \Gamma((a])$ for each $a \in P$.

Lemma 3.2. *Let P be a* PCS *and $a \in P$. Then*

$$\Gamma_a = \Gamma \cap ((a] \times (a]).$$

If in addition $a \in \mathrm{S}(P)$, then $D((a]) = [a]\Gamma$.

Proof. Denote by $\gamma = \Gamma \cap ((a] \times (a])$. Assume that $(c, d) \in \gamma$. Therefore, $c^* = d^*$ and $c, d \in (a]$. Hence, $c^+ = d^+$. Thus, $(c, d) \in \Gamma_a$. Conversely, let $(c, d) \in \Gamma_a$, that means, $c^* \wedge a = d^* \wedge a$ by Lemma 3.1 and $c, d \in (a]$. Evidently, $c^* \wedge a = d^* \wedge a$ implies

$$c^* \wedge a^{**} =^{(4)} c^{***} \wedge a^{**} =^{(5)} (c^* \wedge a)^{**} = (d^* \wedge a)^{**} =^{(5)} d^{***} \wedge a^{**} =^{(4)} d^* \wedge a^{**}$$

in $\mathrm{S}(P)$. We claim that $c^* = d^*$. Really, as $c^*, d^* \geq a^*$ and $\mathrm{S}(P)$ is a Boolean algebra, we get

$$\begin{aligned} c^* &= c^* \sqcup a^* = (c^* \sqcup a^*) \wedge (a^* \sqcup a^{**}) = (c^* \wedge a^{**}) \sqcup a^* \\ &= (d^* \wedge a^{**}) \sqcup a^* = (d^* \sqcup a^*) \wedge (a^* \sqcup a^{**}) = d^* \sqcup a^* = d^*, \end{aligned}$$

as claimed. It follows that $(c, d) \in \Gamma$ and consequently, $(c, d) \in \gamma$. Eventually, $\Gamma_a = \gamma$, as required. The second part of the assertion is obvious. $\square$

Lemma 3.3. *Let P_1 and P_2 be p-semilattices. Then $P_1 \times P_2$ is a p-semilattice such that*

(i) $D(P_1 \times P_2) = D(P_1) \times D(P_2)$,
(ii) $\mathrm{S}(P_1 \times P_2) = \mathrm{S}(P_1) \times \mathrm{S}(P_2)$ *and*
(iii) $\Gamma(P_1 \times P_2) = \Gamma(P_1) \times \Gamma(P_2)$.

Proof. The proof is routine. Recall that $\Theta_1 \times \Theta_2$ is the relation on $P_1 \times P_2$ defined by rule

$$(x_1, x_2) \equiv (y_1, y_2)\ (\Theta_1 \times \Theta_2) \ \text{ iff } \ x_1 \equiv y_1\ (\Theta_1) \ \text{ and } \ x_2 \equiv y_2\ (\Theta_2).$$

$\square$

Lemma 3.4. *Let $a \in P$ for some* PCS *P. Then*

$$\mathrm{S}(a) = \{x^{**} \wedge a : x \in (a]\}.$$

If in addition $a \in \mathrm{S}(P)$, then $\mathrm{S}(a) = (a] \cap \mathrm{S}(P)$.

Proof. Using the notation of Lemma 3.1 we have only to show that

$$\mathrm{S}(a) = \{x^{++} : x \in (a]\}.$$

The rest of the proof follows from Lemma 3.1. $\square$

Theorem 3.5. *Let P be a* PCS. *Then $c \equiv d\ (\Gamma)$ implies $\mathrm{S}(c) \cong \mathrm{S}(d)$. Moreover, $\mathrm{S}(c) = \mathrm{S}(d)$ iff $c = d$.*

Proof. First assume that $c = 1$. Therefore, $d \in [1]\Gamma$. Consider the mapping

$$\varphi : \mathrm{S}(1) \to \mathrm{S}(d)$$

defined by

$$\varphi : x \mapsto x \wedge d.$$

Recall that $x \in \mathrm{S}(1)$ iff x is closed, i.e. $x = x^{**}$. By Lemma 3.4, φ is correctly defined. Now, we wish to show that φ is an isomorphism of Boolean algebras. It suffices to show that it preserves meets and complements. It is immediate from

$$\varphi(x^*) = x^* \wedge d = (\varphi(x))^+ = (x \wedge d)^+.$$

According to Lemma 3.1 we get by (5)

$$(x \wedge d)^+ = (x \wedge d)^* \wedge d = ((x \wedge d)^{**})^* \wedge d = (x^{**} \wedge d^{**})^* \wedge d = x^* \wedge d,$$

as $x \in \mathrm{S}(L)$ and $d^{**} = 1$. Hence φ is a Boolean homomorphism. Again by Lemma 3.4 we see that φ is an epimorphism. It remains to demonstrate that φ is one-to-one. Since φ is a Boolean homomorphism, it is enough to know $\mathrm{Ker}\varphi = \{x : \varphi(x) = 0\}$. Now, suppose that $\varphi(x) = x \wedge d = 0$. It follows that

$$(x \wedge d)^{**} = x^{**} \wedge d^{**} = x^{**} = 0,$$

as $d \in [1]\Gamma$. Hence, $x = 0$. Eventually, $\mathrm{Ker}\varphi = \{0\}$ and φ is an isomorphism of Boolean algebras.

Just we have shown that $\mathrm{S}(1) \cong \mathrm{S}(d)$, whenever $d \in [1]\Gamma$. Take $c, d \in [1]\Gamma$. Since $\mathrm{S}(1) \cong \mathrm{S}(c)$ and $\mathrm{S}(1) \cong \mathrm{S}(d)$, we get $\mathrm{S}(c) \cong \mathrm{S}(d)$.

Now, assume that $c \equiv d\ (\Gamma)$ with $c^{**} = d^{**} = w < 1$. Consider $(w]$, the new PCS, and the Glivenko congruence Γ_w on it (Lemma 3.1). Evidently, $[w]\Gamma_w$ is the filter of dense elements of $(w]$ (Lemma 3,2). In the first part of this proof we have demonstrated $\mathrm{S}(c) \cong \mathrm{S}(d)$, whenever c and d are dense elements. Therefore, this result is true for $c, d \in [w]\Gamma_w$. Since

$$[w]\Gamma = [w]\Gamma_w,$$

by Lemma 3.2, we have $\mathrm{S}(c) \cong \mathrm{S}(d))$, for $c \equiv d(\Gamma)$.

The last part of theorem is obvious and the proof is finished. $\square$

Since $(x, x^{**}) \in \Gamma$ implies $\mathrm{S}(x) \cong \mathrm{S}(x^{**})$, we easily derive

Corollary 3.6. $\mathrm{S}(c) \cong \mathrm{S}(d)$ *iff* $\mathrm{S}(c^{**}) \cong \mathrm{S}(d^{**})$, *for any elements* c, d.

Corollary 3.7. *Let P be a finite* PCS *and let $c, d \in P$. Assume that the elements c^{**} and d^{**} dominate in* $\mathrm{S}(P)$ *i and j atoms, respectively. Then* $\mathrm{S}(c^{**}) \cong \mathrm{S}(d^{**})$ *iff $i = j$.*

4. The spectra

There is an alternative approach to spectra (see Theorem 1.1): they can be defined using the Glivenko congruence. First we present a construction of some finite pseudocomplemented lattices.

Suppose we have a finite non-trivial Boolean lattice B. Moreover, let $\{F_a : a \in B\}$ be a family of finite lattices, which are mutually disjoint. Since F_a, $a \in B$, is finite, it has the largest element 1_a and the smallest one 0_a. In order to avoid intricate notation, we shall write mostly 1 and 0 instead of 1_a and 0_a, respectively. Now, assume additionally that $F_0 = \{0\}$. We construct a new finite lattice $L = (L; \sqsubseteq)$ as follows. Let $L = \{(a, u) : a \in B \text{ and } u \in F_a\}$, and define $(a, u) \sqsubseteq (b, v)$ iff either $a = b$ and $u \leq v$, or $a < b$.

Note that $\sqsubseteq$ is the lexicographic order on L and that $(0,0) \sqsubseteq (a,u) \sqsubseteq (1,1)$ for each $(a,u) \in L$. More information we have from the following

Lemma 4.1. *Let L be the finite bounded poset constructed from a Boolean lattice B and the family of finite lattices $\{F_a : a \in B\}$ as given above. Then L is a finite* PCL *such that* $\mathrm{S}(L) \cong B$ *and* $F_a \cong [(a,1)]\Gamma$ *for each* $a \in B$.

Proof. First we prove that L is a lattice. Assume that $(a,u),(b,v) \in L$. It is easy to show that $(a,u) \sqcap (b,v) = (a \wedge b, w)$ and $(a,u) \sqcup (b,v) = (a \vee b, z)$ where

$$w = \begin{cases} u \wedge v & \text{if } a = b \\ u & \text{if } a < b \\ v & \text{if } b < a \\ 1 \in F_{a\wedge b} & \text{if } a \parallel b \end{cases} \quad \text{and} \quad z = \begin{cases} u \vee v & \text{if } a = b \\ v & \text{if } a < b \\ u & \text{if } b < a \\ 0 \in F_{a\vee b} & \text{if } a \parallel b\ . \end{cases}$$

Therefore, L is a finite lattice. We claim that L is pseudocomplemented with

$$(a,u)^* = (a',1)$$

for arbitrary $(a,u) \in L$. Evidently, $(0,0)^* = (1,1)$ and $(1,1)^* = (0,0)$ as $|F_0| = 1$. Now, assume that $0 < a < 1$ in B. Therefore,

$$(a,u) \sqcap (a',1) = (a \wedge a', w) = (0,0),$$

again as $F_0 = \{0\}$. Suppose that $(a,u) \sqcap (b,v) = (0,0)$ in L. Since,

$$(a,u) \sqcap (b,v) = (a \wedge b, w),$$

we see that $a \wedge b = 0$. Hence, $(b,v) \sqsubseteq (a',1)$ and L is a PCL.

It is easy to verify that $\mathrm{S}(L) = \{(a,1) : a \in B\}$. Moreover, it is not difficult to prove that the mapping $\varphi : B \to \mathrm{S}(L)$ defined by $\varphi : a \mapsto (a,1)$ is a Boolean isomorphism. Thus, $\mathrm{S}(L) \cong B$.

The last statement follows from the fact that $[(a,1)]\Gamma = \{(a,u) : u \in F_a\}$ and the proof is complete. $\square$

Theorem 4.2. *Let L be a finite* PCL *with* $\mathrm{speclength}(L) = n \geq 1$. *Suppose that A_i denote the set of all elements in* $\mathrm{S}(L)\}$ *that are the joins of exactly $i \leq n$ atoms in* $\mathrm{S}(L)$. *Set*

$$s_i = \sum(\ |\ [a]\Gamma\ |\ : a \in A_i)$$

for all $i = 1, \ldots, n$. Then

$$\mathrm{spec}(L) = (1, s_1, \ldots, s_n).$$

Proof. First consider the set

$$C_i = \bigcup([a]\Gamma : a \in A_i) \subseteq L$$

for $1 \leq i \leq n$. Let B_i denote the finite Boolean lattice with i atoms. It is easy to verify by Theorem 3.5 and, Corollaries 3.6 and 3.7 that

$$\mathrm{S}(x) \cong B_i \text{ iff } x \in C_i.$$

Therefore,

$$s_i = \ |C_i| \ = \ \sum(\ |\ [a]\Gamma\ |\ : a \in A_i),$$

as different congruence classes of Γ are disjoint. The proof is finished. $\square$

Now we can present an alternative proof of Theorem 1.1.

Proof of Theorem 1.1. Necessity. Let the sequence $\mathbf{s} = (1, s_1 \ldots, s_n)$ be the spectrum of a finite PCL L. Suppose that A_i denote the set of all elements $\{a \in \mathrm{S}(L)\}$ which are a join of exactly i atoms in $\mathrm{S}(L)$, where $1 \leq i \leq n$. Observe that $|A_i| = \binom{n}{i}$. Theorem 4.2 yields $s_i \geq \binom{n}{i}$ for every $1 \leq i \leq n$.

Sufficiency. Let $\mathbf{s} = (1, s_1, \ldots, s_n)$ be a sequence of positive integers such that $s_i \geq \binom{n}{i}$ for every $1 \leq i \leq n$. Now we shall apply Lemma 4.1. Let us take the Boolean lattice $B = B_n$. In addition, consider the family of finite lattices $\{F_a : a \in B\}$, which are mutually disjoint, $|F_0| = 1$ and

$$s_i = \sum(\ |F_c| : c \in A_i)$$

for $1 \leq i \leq n$. This is possible, because we can choose for F_a suitable chains. By Lemma 4.1, there exists a finite PCL L such that $\mathrm{S}(L) \cong B$ and $[(a, 1)]\Gamma \cong F_a$ for every $a \in B$. It is not difficult to verify that the sequence $\mathbf{s}$ is the spectrum of L, (see Theorem 4.2) and the proof is complete. $\square$

Corollary 4.3. *Let L be a finite* PCL *with* $\mathrm{spec}(L) = (1, s_1, \ldots, s_n)$. *Then L is a Boolean lattice iff*

$$s_i = \binom{n}{i}$$

for every $1 \leq i \leq n$.

Proof. It is well known that L is a Boolean algebra iff the Glivenko congruence $\Gamma = \omega$, i.e. Γ is identity relation (Corollary 2.3). The rest of the proof follows from Theorems 4.2 and 4.3. $\square$

5. Stone lattices

In this section we shall characterize the spectra of finite Stone lattices. First we need some definitions and results.

Definition 5.1. *We say that a distributive p-algebra is a* Stone algebra*, if it satisfies the following Stone identity*

$$x^* \vee x^{**} = 1.$$

The corresponding PCL *is called a* Stone lattice*.*

For more details about distributive PCL-s or Stone lattices we recommend the reader to consult [4], [5] or [9].

The proofs of the following five statements are either routine or they can be found in the books [4] and [5] (see also [8]).

Lemma 5.2. *Let L be a modular p-algebra. Then L satisfies the following identity*

$$x = x^{**} \wedge (x \vee x^*).$$

Lemma 5.3. *Let L be a distributive p-algebra. Then the following conditions are equivalent:*

(i) *L is a Stone algebra,*
(ii) *$(x \wedge y)^* = x^* \vee y^*$ for $x, y \in L$,*
(iii) *$x, y \in \mathrm{S}(L)$ implies that $x \vee y \in \mathrm{S}(L)$ and*
(iv) *$(x \vee y)^{**} = x^{**} \vee y^{**}$ for $x, y \in L$.*

Lemma 5.4. *A finite distributive lattice is a Stone lattice iff it is the direct product of finite distributive dense lattices, that is, finite distributive lattices with only one atom. Moreover, this resulting finite decomposition is unique up to a permutation of factors.*

Lemma 5.5. *Let L be a Stone lattice and let $a \in L$. Then $(a]$ is again a Stone lattice.*

Lemma 5.6. *Let L be a finite distributive dense lattice. Then L is a Stone lattice satisfying*

(i) *$\mathrm{S}(L) = \{0, 1\}$ and*
(ii) *$D(L) = L - \{0\}$.*

We shall work with Stone lattices L, which are written in the form of direct product of dense Stone lattices $L_1, \dots, L_n$ (see Lemma 5.4), that means

$$L = L_1 \times \cdots \times L_n = \prod (L_i : 1 \leq i \leq n).$$

Lemma 5.7. *Let L be a finite Stone lattice, which is the direct product of dense Stone lattices $L_1, \dots, L_n$ for some $n \geq 1$. Then*

$$\mathrm{S}(L) = \mathrm{S}(L_1) \times \cdots \times \mathrm{S}(L_n) = \{(a_1, \dots, a_n) \in L : a_j \in \{0, 1\} \ \text{ for every } \ 1 \leq j \leq n\}.$$

Proof. The proof is straightforward (see Lemma 3.3). □

Next we shall describe the Glivenko congruence $\Gamma(L)$ of finite Stone lattices $L = \prod (L_i : 1 \leq i \leq n)$, where L_i, $1 \leq i \leq n$, are dense Stone lattices. It is enough to know the subsets

$$[b]\Gamma(L) = F_b,$$

for every $b \in \mathrm{S}(L)$. Clearly, the set $\{F_b : b \in \mathrm{S}(L)\}$ forms a partition of L. We shall use a new notation for the elements of $\mathrm{S}(L) - \{0\}$. If $1 \leq i_1 < \cdots < i_k \leq n$, and $1 \leq k \leq n$, then we set $e(i_1, \dots, i_k) = (a_1, \dots, a_n) \in \mathrm{S}(L) - \{0\}$ where $a_j = 1$ for $j \in \{i_1, \dots, i_k\}$ and $a_j = 0$ for $j \notin \{i_1, \dots, i_k\}$. In particular, $e(i) = (0, \dots, 0, 1, 0 \dots, 0)$ with 1 being in the i-th position.

Now, we can formulate the first result.

Lemma 5.8. *Under the hypothesis of Lemma 5.7 we have:*

(i) *The elements $e(1), \dots, e(n)$ are all atoms of $\mathrm{S}(L)$.*
(ii) *$e(i_1, \dots, i_k) = e(i_1) \vee \cdots \vee e(i_k)$ in $\mathrm{S}(L)$ for any $1 \leq i_1 < \cdots < i_k \leq n$.*
(iii) *$(e(i_1, \dots, i_k)]$ and $L_{i_1} \times \cdots \times L_{i_k}$ are isomorphic as p-algebras.*
(iv) *$F_{e(i_1, \dots, i_k)} \cong D(L_{i_1} \times \cdots \times L_{i_k}) = D(L_{i_1}) \times \cdots \times D(L_{i_k})$.*

Proof. Parts (i) and (ii) follow from Lemmas 5.3 and 5.7. To show (iii), observe that mapping

$$\varphi : L_{i_1} \times \cdots \times L_{i_k} \to (e(i_1, \ldots, i_k)]$$

defined by

$$\varphi : (x_{i_1}, \ldots, x_{i_k}) \mapsto (a_1, \ldots, a_n),$$

where $a_{i_j} = x_{i_j}$ for $1 \leq j \leq k$ and $a_l = 0$ for $l \notin \{i_1, \ldots, i_k\}$, is a lattice isomorphism. In addition, since both lattices are pseudocomplemented (Lemma 3.1), we see that $L_{i_1} \times \cdots \times L_{i_k}$ and $(e(i_1, \ldots, i_k)]$ are isomorphic as p-algebras. To establish part (iv), observe that

$$D((a]) = [a]\Gamma_a = [a]\Gamma(L) = F_a,$$

for $a = e(i_1, \ldots, i_k)$, by Lemma 3.2. Notice that by (iii) and Lemma 3.3,

$$D((e(i_1, \ldots, i_k)]) \cong D(L_{i_1} \times \cdots \times L_{i_k}) = D(L_{i_1}) \times \cdots \times D(L_{i_k}),$$

which is just the desired result and the proof is finished. □

We need the following (known) concept: As usual,

$$\sigma_k(x_1, \ldots, x_n) = \sum(\prod(x_i : i \in H) : H \subseteq \{1, \ldots, n\},\ |H| = k)$$

will stand for the k-th n-variate elementary symmetric polynomial.

It is convenient now to state the result towards which we are aiming:

Theorem 5.9. *Let $L_1, \ldots, L_n$ be finite dense Stone lattices. Consider the Stone lattice $L = L_1 \times \cdots \times L_n$. Set $s_k = \sigma_k(|D(L_1)|, \ldots, |D(L_n)|)$. Then $\boldsymbol{s} = (1, s_1, \ldots, s_n)$ is the spectrum of L.*

Proof. For $k \in \{1, \ldots, n\}$, let $M_k = \bigcup(F_{e(i_1,\ldots,i_k)} : 1 \leq i_1 < \cdots < i_k \leq n)$. Since

$$|F_{e(i_1,\ldots,i_k)}| = |D(L_{i_1})| \ldots |D(L_{i_k})|$$

by Lemma 5.8 and the above union is a disjoint one, we conclude that

$$|M_k| = \sigma_k(|D(L_1)|, \ldots, |D(L_n)|) = s_k.$$

Each $a \in L$ belongs to a unique $F_{e(j_1,\ldots,j_m)}$, and then $S(a) \cong B_m$ by Lemma 3.4 and Theorem 3.5. Hence the k-th entry of the spectrum of L is $|M_k| = s_k$, indeed. □

Corollary 5.10. *Let L be a finite Stone lattice with the spectrum $\boldsymbol{s} = (1, s_1, \ldots, s_n)$. Then L is a Boolean lattice iff $s_1 = n$.*

Proof. If L is a Boolean lattice, then $s_1 = n$ by Corollary 4.3. Conversely, assume that $s_1 = n$. Theorem 5.9 implies that

$$t_1 + \cdots + t_n = s_1 = n$$

for $t_i = |D(L_i)|$, $i = 1, \ldots, n$. Therefore, $t_1 = \cdots = t_n = 1$. Thus, by Theorem 5.9, we have $s_i = \binom{n}{i}$ for every $i = 1, \ldots, n$. Again applying Corollary 4.3 we can conclude that L is a Boolean lattice. □

Now, we can formulate the main result of this section. Observe that the following theorem is a partial solution of Problem 1 from [6].

Theorem 5.11. *Let $n \geq 1$ be an integer and assume that $\mathbf{c} = (1, c_1, \ldots, c_n)$ is a sequence of some positive integers. Then $\mathbf{c}$ is the spectrum of a finite Stone lattice iff there exist positive integers $p_1, \ldots, p_n$ such that*

$$c_k = \sigma_k(p_1, \ldots, p_n)$$

for every $1 \leq k \leq n$.

Proof. If L is a finite Stone lattice, then its spectrum satisfies the desired condition by Lemma 5.4 and Theorem 5.9.

Conversely, assume that $\mathbf{c} = (1, c_1, \ldots, c_n)$ satisfies the condition of the Theorem. For each $i \in \{1, \ldots, n\}$, take a finite distributive lattice D_i consisting of exactly p_i elements. Adding a new zero element to D_i, we get a distributive dense lattice L_i. It is a Stone lattice and $|D(L_i)| = |D_i| = p_i$ by Lemma 5.6. Finally, c is the spectrum of the Stone lattice $L = L_1 \times \cdots \times L_n$ by Theorem 5.9. □

6. Closing remarks

It is interesting to observe that the two sequences $(1, k)$ and $(1, k+1, k)$ mentioned in [6] are representing spectra of two finite Stone lattices K and M, respectively. It is easy to show that K is a distributive dense lattice with $|D(K)| = k$. (Dropping distributivity we get a more general situation: A finite PCL L has $\mathrm{spec}(L) = (1, k)$ iff L is a dense lattice with $|D(L)| = k$.) In the second event, put $t_1 = k$ and $t_2 = 1$. Applying Theorem 5.15, we see that $c_1 = k + 1$ and $c_2 = k$. Now, M is a direct product of two dense distributive lattices L_1 and L_2 defined as follows: $D(L_1)$ is distributive lattice with k elements and $|L_2| = 2$.

Furthermore the second remark concerns the sequence $(1, 3, 1)$ also mentioned in [6]. We can present the following result:

Proposition 6.1. *Let L be a finite modular pseudocomplemented lattice with spectrum $\boldsymbol{s} = (1, s_1, \ldots, s_{n-1}, 1)$. Then $s_i = \binom{n}{i}$ for $i = 0, 1, \ldots, n$, and L is a Boolean lattice.*

Proof. Consider the mapping $\varphi : F_a \to D(L)$, for arbitrary $a \in \mathrm{S}(L)$, defined by

$$\varphi : x \mapsto x \vee x^* = x \vee a^*.$$

For more details see 3.1 of [8] or [10]. Since $(x \vee x^*)^* = 0$, $\varphi(x)$ is really in $F_1 = D(L)$. If $x, y \in F_a$ with $\varphi(x) = \varphi(y)$, then, by Lemma 5.2,

$$x = x^{**} \wedge (x \vee x^*) = a \wedge \varphi(x) = a \wedge \varphi(y) = y^{**} \wedge (y \vee y^*) = y.$$

Hence φ is injective. Since $|D(L)| = |F_1| = s_n = 1$ by the assumption, the injectivity of φ yields that $|F_a| = 1$, for all $a \in \mathrm{S}(L)$. Hence $\Gamma = \omega$, whence $L = \mathrm{S}(L)$ and the argument for Corollary 4.3 applies. Finally, since $\mathrm{S}(L)$ is Boolean, so is L. □

Corollary 6.2. *There is no modular* PCL *with spectrum* $(1, 3, 1)$.

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References

[1] Birkhoff, G.: Lattice Theory, third edn. Amer. Math. Soc., Providence, R.I. (1967)
[2] Frink, O.: Pseudo-complements in semi-lattices. Duke Math. J. **29**, 505-514 (1962)
[3] Glivenko, V.: Sur quelques points de la logique de M. Brouwer. Bull Acad. Sci. Belgique **15**, 183-188 (1929)
[4] Grätzer, G.: Lattice Theory. First Concepts and Distributive Lattices. W. H. Freeman and Co., San Francisco (1971)
[5] Grätzer, G.: General Lattice Theory, 2nd edn. New appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung, and R. Wille. Birkhäser Verlag, Basel (1998)
[6] Grätzer, G., Gunderson, D. S., Quackenbush, R., W.: The spectrum of a finite pseudocomplemented lattice. Algebra Universalis, **61**, 407-411 (2009)
[7] Jacobson, N.: Lectures in Abstract Algebra, Vol. I, D. van Nostrand Comp., Inc. Princeton, N.J. (1964)
[8] Katriňák, T.: Subdirectly irreducible modular p-algebras. Algebra Universalis **2**, 166-173 (1972)
[9] Katriňák, T.: A new proof of the construction theorem for Stone algebras. Proc. Amer. Math. Soc. **40**, 75-78 (1973)
[10] Katriňák, T., Mederly, P.: Construction of modular p-algebras. Algebra Universalis **4**, 301-315 (1974)

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