COMPLEMENTED P-ALGEBRAS

TIBOR KATRINAK, Bratislava

An algebra $L = (L; \lor, \land, *, 0, 1)$ is called a $p$-algebra (or a pseudocomplemented lattice) if $(L; \lor, \land, 0, 1)$ is a bounded lattice and $x \leq a^*$ iff $a \land x = 0$. A $p$-algebra is said to be nontrivial if $x^* \neq 0$ whenever $x \neq 1$.

It is known in Lambrou [4] using the axiom of choice and working with ultrafilters that a nontrivial $p$-algebra is complemented. We give a short elementary proof of this fact (Theorem 1). In special cases complemented $p$-algebras are Boolean algebras. In Theorem 2 we characterize those equational classes of $p$-algebras in which that is the case.

**Theorem 1.** A $p$-algebra $L$ is complemented if and only if $L$ is nontrivial.

**Proof.** Since $x^* \leq x^*$, we see that a complemented $p$-algebra is nontrivial. Conversely, let $L$ be nontrivial. Evidently, $x \leq y$ yields $x^* \geq y^*$. Therefore, $(x \lor x^*)^* \leq x^* \land x^{**} = 0$, and this implies $x \lor x^* = 1$. Hence $x^*$ is a complement of $x$ in $L$.

Pentagon, the five-element nonmodular lattice, is the smallest non-Boolean complemented $p$-algebra. It is known that the $p$-algebras can be defined in terms of identities, that means the class of all $p$-algebras is equational (cf. [1]).

**Theorem 2.** Let $K$ be an equational subclass of the class of all $p$-algebras. The following conditions are equivalent:

(i) $K$ does not contain a pentagon;

(ii) Every algebra from $K$ satisfies identity

$$x = x^{**} \land (x \lor x^*)$$

(iii) In $K$, every complemented algebra is a Boolean one.

**Proof.** (i) $\Rightarrow$ (ii). In [2, Theorems 4 and 6] we have shown that the equational subclasses satisfying (i) are contained in the class of $p$-algebras defined by identity $x = x^{**} \land (x \lor x^*)$. (ii) $\Rightarrow$ (iii). Since $x \lor x^* = 1$ in a complemented $p$-algebra,
we see that $x = x^{**}$ for every $x$. But the "closed" elements form a Boolean algebra (see [1, Theorem 6.4]). (iii) $\Rightarrow$ (i) is trivial.

Remark. In view of Theorem 2 we can give a different proof of Theorem in [3]: A complete lattice with 0 and 1 is an atomic Boolean algebra if and only if it is semisimple (i.e. the intersection of all maximal ideals of $L$ is 0) and completely distributive. Evidently, $L$ is distributive, pseudocomplemented and dually pseudocomplemented, by complete distributivity. (Let $a^\ast$ denote the dual pseudocomplement of $a$, i.e. $a \lor x = 1$ iff $x \geq a^\ast$.) Then $x = x^{**} \lor (x \land x^\ast)$ and, by [1, Lemma 15.5], every maximal ideal of $L$ contains ideal $\{x \in L : x^\ast = 1\} = \{x \land x^\ast : x \in L\}$. By semisimplicity, $x \land x^\ast = 0$ for every $x \in L$. Hence $L$ is complemented. Therefore, $L$ is a Boolean algebra by Theorem 2. The rest follows from the Tarski’s Theorem for Boolean algebras.

REFERENCES


Received: 18. 10. 1982

Tibor Katriňák
Katedra algebry a teorie čísel MFF UK
matematický pavilon
Mlynská dolina
842 15 Bratislava

SÚHRN

KOMPLEMENTÁRNE $p$-ALGEBRY
T. Katriňák, Bratislava

V práci sa charakterizujú tie $p$-algeby, ktoré sú aj komplementárnymi zväzmi.

РЕЗЮМЕ

$p$-АЛГЕБРЫ С ДОПОЛНЕНИЯМИ
T. Катриняк, Братислава

Охарактеризованы те $p$-алгебры, которые являются решеткой с дополнениями.