ON SOME PROPERTIES OF $\mathcal{I}$-CONVERGENCE

TIBOR ŠALÁT, BINOD CHANDRA TRIPATHY AND MILOŠ ZIMAN

ABSTRACT. In the paper [5] the concept of $\mathcal{I}$-convergence is introduced. This concept is a generalization of the statistical convergence. In this paper some notions and results from the statistical convergence are extended to the $\mathcal{I}$-convergence.

INTRODUCTION

The notion of a statistical convergence was introduced by Fast [1] and Schoenberg [6] independently. Later on it was further investigated from a sequence space point of view and linked with the Summability Theory by Fridy [3], Šalát [7], Tripathy [9, 10] and many others.

The notion of an $\mathcal{I}$-convergence is a generalization of the statistical convergence. It was introduced by Kostyrko, Šalát and Wilczyński [5]. Some further results connected with the notion of the $\mathcal{I}$-convergence can be found in [4] and [8].

In this papers we extend some notions and results known for the statistical convergence to the $\mathcal{I}$-convergence.

1. DEFINITIONS AND PRELIMINARIES

Throughout the article w, c, c_0, ℓ_∞ denote the spaces of *all, convergent, null* and *bounded complex valued sequences*, respectively.

Definition 1.1. A non-void class $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an *ideal* if $\mathcal{I}$ is additive (i.e., $A,\ B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$) and hereditary (i.e., $A \in \mathcal{I} \,\&\, B \subseteq A \Rightarrow B \in \mathcal{I}$).

An ideal $\mathcal{I}$ is said to be *non-trivial* if $\mathcal{I} \neq 2^{\mathbb{N}}$.

A non-trivial ideal $\mathcal{I}$ is said to be *admissible* if $\mathcal{I}$ contains every finite subset of $\mathbb{N}$.

All ideals in this paper are assumed to be admissible.

For any ideal there is a filter $\mathcal{F}(\mathcal{I})$ corresponding with $\mathcal{I}$, given by

$$\mathcal{F}(\mathcal{I}) = \{K \subseteq \mathbb{N} : \mathbb{N} \smallsetminus K \in \mathcal{I}\}.$$

Definition 1.2. A sequence $x = (x_n) \in w$ is said to be *$\mathcal{I}$-convergent* if there exists $L \in \mathbb{C}$ such that for all $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}$. We say, x *$\mathcal{I}$-converges to* L, and write $\mathcal{I}\text{-}\lim x = L$. The number L is called *$\mathcal{I}$-limit of* x (cf. [5]).

The usual convergence is a special case of $\mathcal{I}$-convergence ($\mathcal{I} = \mathcal{I}_f$ - the ideal of all finite subsets of $\mathbb{N}$). The *statistical convergence* (see [1] or [6]) is also the special case of $\mathcal{I}$-convergence. In this case $\mathcal{I} = \mathcal{I}_d = \left\{A \subseteq \mathbb{N} : \lim\limits_{n\to\infty} \frac{|A \cap \{1,2,\ldots,n\}|}{n} = 0\right\}$,

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2000 *Mathematics Subject Classification.* 40A05.

Key words and phrases. density, solid space, $\mathcal{I}$-convergence, symmetric space.

$|M|$ being the cardinality of the set M. For other examples of $\mathcal{I}$-convergence see [5].

Remark. One can readily see that the $\mathcal{I}$-convergence of sequences of points in a metric space can be defined analogously.

The class of all $\mathcal{I}$-convergent sequences is denoted by $c^{\mathcal{I}}$ and $\mathcal{I}$-null sequences (i.e, $\mathcal{I}\text{-}\lim x = 0$) by $c_0^{\mathcal{I}}$.

Definition 1.3. A sequence $(x_n) \in w$ is said to be:

- *$\mathcal{I}^*$-convergent to L* ($\mathcal{I}^*\text{-}\lim_{n\to\infty} x_n = L$) if there is a set $\{n_1 < n_2 < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that $\lim_{i\to\infty} x_{n_i} = L$;
- *$\mathcal{I}$-bounded* if there exists an $M > 0$ such that $\{k \in \mathbb{N} : |x_k| > M\} \in \mathcal{I}$.

2. Symmetry and monotonicity of $c^{\mathcal{I}}$

Definition 2.1. A set $E \subseteq w$ is said to be:

- *solid* if $(\alpha_n x_n) \in E$ whenever $(x_n) \in E$, $(\alpha_n) \in w$ and $|\alpha_n| \le 1$ for every $n \in \mathbb{N}$;
- *symmetric* if $(x_{\pi(n)}) \in E$ whenever $(x_n) \in E$ and π is a permutation of $\mathbb{N}$.

Definition 2.2. Let $K = \{k_1 < k_2 < \dots\} \subseteq \mathbb{N}$ and E be a sequence space. A *K-step space of E* is a sequence space $\lambda_K^E = \{(x_{k_n}) \in w : (x_n) \in E\}$.

A *canonical preimage of a sequence* $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_n) \in w$ defined as follows:

$$y_n = \begin{cases} x_n, & \text{if } n \in K; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

A *canonical preimage of a step space* λ_K^E is a set of canonical preimages of all elements in λ_K^E, i.e., y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

Definition 2.3. A sequence space E is said to be *monotone* if it contains the canonical preimage of all its step spaces (i.e., if for all infinite $K \subseteq \mathbb{N}$ and $(x_n) \in E$ the sequence $(\alpha_n x_n)$, where $\alpha_n = 1$ for $n \in K$ and $\alpha_n = 0$ otherwise, belongs to E).

The following proposition is obvious.

Proposition 2.4. *Every solid sequence space is monotone.*

The following lemmas will be used for establishing some results of the present article.

Lemma 2.5 (Kostyrko, Šalát and Wilczyński [5], Proposition 3.2)**.** *Let $\mathcal{I}$ be an admissible ideal. If $\mathcal{I}^*\text{-}\lim x = L$ then $\mathcal{I}\text{-}\lim x = L$.*

Lemma 2.6 (Šalát, Tripathy and Ziman [8], Lemma 2.5)**.** *Let $K \in \mathcal{F}(\mathcal{I})$ and $M \subseteq \mathbb{N}$. If $M \notin \mathcal{I}$ then $M \cap K \notin \mathcal{I}$.*

Theorem 2.7. *If $\mathcal{I}$ is not maximal, then the space $c^{\mathcal{I}}$ is neither solid nor monotone.*

Proof. By Proposition 2.4 it is enough to show that $c^{\mathcal{I}}$ is not monotone.

Let $x_n = 1$ for all $n \in \mathbb{N}$. Then obviously $(x_n) \in c^{\mathcal{I}}$. And let $K \subseteq \mathbb{N}$ be such that $K \notin \mathcal{I}$ and $\mathbb{N} \setminus K \notin \mathcal{I}$.

Define the sequence (y_n) by (1). Then (y_n) belongs to canonical preimage of K-step space of $c^{\mathcal{I}}$, but $(y_n) \notin c^{\mathcal{I}}$. Hence $c^{\mathcal{I}}$ is not monotone. □

Remark. By [8, Proposition 4.3] $c^{\mathcal{I}}$ is solid for every maximal ideal $\mathcal{I}$.

Remark. By [8, Proposition 3.9] $c_0^{\mathcal{I}}$ is solid for every non-maximal ideal. The case of maximal ideal can be treated similarly as in [8, Proposition 4.3].

Theorem 2.8. *If $\mathcal{I}$ is not maximal and $\mathcal{I} \neq \mathcal{I}_f$, then $c^{\mathcal{I}}$ and $c_0^{\mathcal{I}}$ are not symmetric.*

Proof. Let $A \in \mathcal{I}$ be infinite. If

$$x_n = \begin{cases} 1, & \text{for } n \in A; \\ 0, & \text{otherwise,} \end{cases}$$

then by Lemma 2.5$(x_n) \in c_0^{\mathcal{I}} \subseteq c^{\mathcal{I}}$.

Let $K \subseteq \mathbb{N}$ be such that $K \notin \mathcal{I}$ and $\mathbb{N} \setminus K \notin \mathcal{I}$. Let $\varphi : K \to A$ and $\psi : \mathbb{N} \setminus K \to \mathbb{N} \setminus A$ be bijections. (All four sets are infinite.) Then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by:

$$\pi(n) = \begin{cases} \varphi(n), & \text{if } n \in K; \\ \psi(n), & \text{otherwise} \end{cases}$$

is a permutation of $\mathbb{N}$, but $(x_{\pi(n)}) \notin c^{\mathcal{I}}$ and $(x_{\pi(n)}) \notin c_0^{\mathcal{I}}$. Hence $c_0^{\mathcal{I}}$ and $c^{\mathcal{I}}$ are not symmetric. □

3. Real sequences

During this section we restrict our attention to real valued sequences. We give some simple but useful remarks about properties of these sequences.

Definition 3.1. A real valued sequence $x = (x_n)$ is said to be *$\mathcal{I}$-monotonic increasing* (*$\mathcal{I}$-monotonic decreasing*), if there is a set $\{k_1 < k_2 < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that $x_{k_i} \leq x_{k_{i+1}}$ ($x_{k_i} \geq x_{k_{i+1}}$) for every $i \in \mathbb{N}$;

Proposition 3.2. *A real sequence $x = (x_n)$ is $\mathcal{I}$-monotonic increasing ($\mathcal{I}$-monotonic decreasing) if and only if x can be written in the form*

$$x = y + z; \tag{2}$$

where $y = (y_n)$ is nondecreasing (nonincreasing) real sequence and $z = (z_n)$ is such that $\{n \in \mathbb{N};\ z_n \neq 0\} \in \mathcal{I}$.

in this case, if y is bounded, then x is $\mathcal{I}$-convergent and $\mathcal{I}\text{-}\lim x = \lim_{n\to\infty} y_n$.

Proof. Assume that x is $\mathcal{I}$-monotonic increasing. The other case can be shoved analogously. Then there exists a set $K = \{k_1 < k_2 < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that $x_{k_i} \leq x_{k_{i+1}}$ for every $i \in \mathbb{N}$. Denote $m_n = \max\{k \in K;\ k \leq n\}$. Now we can define the sequences (y_n), (z_n) as follows:

$$y_n = x_{m_n}, \qquad n \in \mathbb{N} \tag{3}$$

$$z_n = \begin{cases} 0, & n \in K; \\ x_n, & \text{otherwise.} \end{cases} \tag{4}$$

Clearly, $x = y + z$, y is nondecreasing and $\{n \in \mathbb{N};\, z_n \neq 0\} \in \mathcal{I}$. The converse implication is obvious.

If $x = y + z$, y, z with described properties, and y is bounded, then it is convergent. It is a routine work to verify that x is $\mathcal{I}$-convergent and $\mathcal{I}\text{-}\lim x = \lim_{n\to\infty} y_n$. □

Corrolary 3.3. *An $\mathcal{I}$-monotonic sequence is $\mathcal{I}$-convergent if and only if it is $\mathcal{I}$-bounded.*

Proof. Let x be $\mathcal{I}$-monotonic increasing and $\mathcal{I}$-convergent sequence. Let $x = y + z$ the decomposition of x from previous proposition. It is clear that $\mathcal{I}\text{-}\lim z = 0$. Hence, as $y = x - z$, y $\mathcal{I}$-convergent and $\mathcal{I}\text{-}\lim y = \mathcal{I}\text{-}\lim x$. Then, it follows from monotonicity of y that y is convergent and so bounded. Thus x is $\mathcal{I}$-bounded.

If x is $\mathcal{I}$-bounded, then the decomposition $x = y + z$ can be made such that y is bounded:

Let $K = \{k_1 < k_2 < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that $x_{k_i} \le x_{k_{i+1}}$ for every $i \in \mathbb{N}$ and $M > 0$ be such that $K' = \{n \in N\ |x_n| \le M\} \in \mathcal{F}(\mathcal{I})$. Denote $m_n = \max\{k \in K \cap K';\, k \le n\}$ and define the sequences (y_n), (z_n) by (3) and (4).

The rest follows from previous proposition. □

Proposition 3.4. *Let $x = (x_n)$, $y = (y_n)$ and $z = (z_n)$ be real sequences such that $x_n \le y_n \le z_n$ for all $n \in K$, where $K \in \mathcal{F}(\mathcal{I})$. If $\mathcal{I}\text{-}\lim x = L = \mathcal{I}\text{-}\lim z$ then $\mathcal{I}\text{-}\lim y = L$.*

Proof. For a given $\varepsilon > 0$, let

$$B_x = \{n \in N;\, |x_n - L| < \varepsilon\},$$
$$B_y = \{n \in N;\, |y_n - L| < \varepsilon\},$$
$$B_z = \{n \in N;\, |z_n - L| < \varepsilon\}.$$

Then $B_x \cap B_z \cap K \subseteq B_y$, and hence $B_y \in \mathcal{F}(\mathcal{I})$. It follows that $\mathcal{I}\text{-}\lim y = L$. □

4. Convergence field of $\mathcal{I}$-convergence and $\mathcal{I}$-limit of a sequence

In this section we will study some properties of the function, which assigns to every $\mathcal{I}$-convergent sequence its $\mathcal{I}$-limit.

Definition 4.1. A *convergence field of $\mathcal{I}$-convergence* is a set $F(\mathcal{I}) = \{x = (x_n) \in \ell_\infty :$ there exists $\mathcal{I}\text{-}\lim x \in \mathbb{R}\}$.

Remark. The convergence field $F(\mathcal{I})$ is a closed linear subspace of ℓ_∞ with respect to sup-norm, $F(\mathcal{I}) = \ell_\infty \cap c^{\mathcal{I}}$ (cf. [5]).

Define a function $g : F(\mathcal{I}) \to \mathbb{R}$:

$$g(x) = \mathcal{I}\text{-}\lim x, \qquad x \in F(\mathcal{I}).$$

Theorem 4.2. *The function $g : F(\mathcal{I}) \to \mathbb{R}$ is Lipschitz function and hence uniformly continuous.*

Proof. Let $x, y \in F(\mathcal{I})$, $x \neq y$. Then the sets

$$A_x = \{n \in \mathbb{N} :\, |x_n - g(x)| \ge \|x - y\|\},$$
$$A_y = \{n \in \mathbb{N} :\, |y_n - g(y)| \ge \|x - y\|\}$$

belong to $\mathcal{I}$. Thus the sets

$$B_x = \mathbb{N} \setminus A_x = \{n \in \mathbb{N} : |x_n - g(x)| < \|x - y\|\},$$
$$B_y = \mathbb{N} \setminus A_y = \{n \in \mathbb{N} : |y_n - g(y)| < \|x - y\|\}$$

belong to $\mathcal{F}(\mathcal{I})$. Hence also $B = B_x \cap B_y \in \mathcal{F}(\mathcal{I})$, so that $B \neq \emptyset$. Now, taking $n \in B$,

$$|g(x) - g(y)| \leq |g(x) - x_n| + |x_n - y_n| + |y_n - g(y)| \leq 3\|x - y\|.$$

This means that g is Lipschitz function. □

Proposition 4.3. *If $x, y \in F(\mathcal{I})$ then $xy \in F(\mathcal{I})$ and $g(xy) = g(x)g(y)$.*

Proof. Let $\varepsilon > 0$. The sets

$$B_x = \{n \in \mathbb{N} : |x_n - g(x)| < \varepsilon\},$$
$$B_y = \{n \in \mathbb{N} : |y_n - g(y)| < \varepsilon\}$$

belong to $\mathcal{F}(\mathcal{I})$. Now,

$$\begin{aligned} |x_n y_n - g(x)g(y)| &= |x_n y_n - x_n g(y) + x_n g(y) - g(x)g(y)| \\ &\leq |x_n||y_n - g(y)| + |g(y)||x_n - g(x)|. \end{aligned} \tag{5}$$

As $F(\mathcal{I}) \subseteq \ell_\infty$, there exists an $M \in \mathbb{R}$ such that $|x_n| < M$ and $|g(y)| < M$.

Thus, using (5), we get

$$|x_n y_n - g(x)g(y)| \leq M\varepsilon + M\varepsilon = 2M\varepsilon$$

for all $n \in B_x \cap B_y$ ($\in \mathcal{F}(\mathcal{I})$). Hence $xy \in F(\mathcal{I})$ and $g(xy) = g(x)g(y)$. □

5. Cauchy condition for $\mathcal{I}$-convergence

In the case $\mathcal{I} = \mathcal{I}_d$ (statistical convergence) J. A. Fridy [2] proved the corresponding Cauchy condition for (statistical) convergence of a sequence. We will generalize this result for $\mathcal{I}$-convergence:

Theorem 5.1. *A sequence $x = (x_n) \in w$ $\mathcal{I}$-converges if and only if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that*

$$\{n \in \mathbb{N} : |x_n - x_{N_\varepsilon}| < \varepsilon\} \in \mathcal{F}(\mathcal{I}). \tag{6}$$

Proof. Let $L = \mathcal{I}\text{-}\lim x$. Then $B_\varepsilon = \{n \in \mathbb{N} : |x_n - L| < \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$ for all $\varepsilon > 0$. Fix an $N_\varepsilon \in B_\varepsilon$. Then the following holds for all $n \in B_\varepsilon$:

$$|x_{N_\varepsilon} - x_n| \leq |x_{N_\varepsilon} - L| + |L - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence (6) holds.

Now, let (6) hold for all $\varepsilon > 0$. Then the set

$$C_\varepsilon = \{n \in \mathbb{N} : x_n \in [x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon]\} \in \mathcal{F}(\mathcal{I})$$

for all $\varepsilon > 0$. Denote $J_\varepsilon = [x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon]$.

Fix an $\varepsilon > 0$. Then $C_\varepsilon \in \mathcal{F}(\mathcal{I})$ and $C_{\frac{\varepsilon}{2}} \in \mathcal{F}(\mathcal{I})$. Hence $C_\varepsilon \cap C_{\frac{\varepsilon}{2}} \in \mathcal{F}(\mathcal{I})$. This implies

$$J = J_\varepsilon \cap J_{\frac{\varepsilon}{2}} \neq \emptyset,$$
$$\{n \in \mathbb{N} : x_n \in J\} \in \mathcal{F}(\mathcal{I}),$$
$$\operatorname{diam}(J) \leq \frac{1}{2}\operatorname{diam}(J_\varepsilon).$$

(diam(I) denotes the length of the interval I.)

This way, by induction, we can construct the sequence of (closed) intervals

$$J_\varepsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

with the property $\mathrm{diam}(I_n) \leq \frac{1}{2}\mathrm{diam}(I_{n-1})$ $(n = 2, 3, \dots)$ and $\{k \in \mathbb{N} : x_k \in I_n\} \in \mathcal{F}(\mathcal{I})$ $(n = 1, 2, \dots)$. Then there exists a $\xi \in \bigcap_{n\in\mathbb{N}} I_n$ and it is a routine work to verify that $\xi = \mathcal{I}\text{-}\lim x$. □

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Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina, 842 48 Bratislava, SLOVAKIA

Mathematical Sciences Division, Institute of Advanced Study in Science and Technology, Khanapara, Guwahati - 718 022, INDIA

E-mail address: `tripathybc@yahoo.com`

Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina, 842 48 Bratislava, SLOVAKIA

E-mail address: `ziman@fmph.uniba.sk`