ON $\mathcal{I}$-CONVERGENCE FIELD

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Abstract. In this paper we introduce the notion of $c_A^{\mathcal{I}}$ and $m_A^{\mathcal{I}}$, the $\mathcal{I}$-convergence field and bounded $\mathcal{I}$-convergence field of an infinite matrix A. We restrict our study to diagonal matrices. We find necessary and/or sufficient conditions on the elements of A for solidity of $c_A^{\mathcal{I}}$ and $m_A^{\mathcal{I}}$.

1. Introduction

This paper is a generalization of previous paper of Maddox (*cf.* [7]). The second of the authors came with an idea to extend the notion of summability fields of a matrix A to statistical convergence field of A (using statistical convergence instead of usual convergence) and the first of the authors noticed the possibility of extension of these notions with the help of $\mathcal{I}$-convergence.

The concept of $\mathcal{I}$-convergence of sequences in a metric space is introduced in [5].

The statistical convergence is a special case of $\mathcal{I}$-convergence. It was introduced in [2] and [11] independently. Later on it was studied from sequence space point of view and linked with summability in [1], [3], [4], [6], [8], [9], [10], [12], [13] and many others.

2. Definitions and Background

Let X be a normed linear space. We denote by $w(X)$, $c(X)$, $c_0(X)$, $l_1(X)$, $l_\infty(X)$ the sets of *all, convergent, null, absolutely summable* and *bounded sequences*, respectively, with elements from X. For $X = \mathbb{C}$ we will write simply w, c, c_0, l_1 and l_∞.

We recall the concept of an ideal of subsets of $\mathbb{N}$. A non-void class $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called *the ideal* if $\mathcal{I}$ is additive (*i.e.*, $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$) and hereditary (*i.e.*, $A \in \mathcal{I}$, $B \subseteq A \Rightarrow B \in \mathcal{I}$). An ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called *non-trivial* if $\mathcal{I} \neq 2^{\mathbb{N}}$.

Definition 2.1. A non-trivial ideal $\mathcal{I}$ of subsets of $\mathbb{N}$ is said to be *admissible* if $\mathcal{I}$ contains every finite subset of $\mathbb{N}$.

Throughout this paper the calligraphic letter $\mathcal{I}$ will denote a non-trivial admissible ideal of subsets of $\mathbb{N}$.

Definition 2.2. Let X be a normed linear space. We say that $x = (x_n) \in w(X)$ is *$\mathcal{I}$-convergent* if there exists $\xi \in X$ such that for all $\varepsilon > 0$ the following holds:

$$\{n \in \mathbb{N} : \|x_n - \xi\| \geq \varepsilon\} \in \mathcal{I}.$$

($\|\cdot\|$ denotes the norm in X.) In this case we say that x *$\mathcal{I}$-converges* to ξ and write $\xi = \mathcal{I}\text{-}\lim_{n\to\infty} x_n$ (*cf.* [5]).

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Example 2.3. (a) The usual convergence is a special case of $\mathcal{I}$-convergence. In this case: $\mathcal{I} = \mathcal{I}_f$ (the ideal of all finite subsets of $\mathbb{N}$).

(b) Taking $\mathcal{I} = \mathcal{I}_d$ ($\mathcal{I}_\delta$), where $\mathcal{I}_d$ ($\mathcal{I}_\delta$) is the ideal of subsets of $\mathbb{N}$ of asymptotic (logarithmic) density zero, we get the statistical convergence (the logarithmic statistical convergence).

(c) For further examples of $\mathcal{I}$-convergence *see* [5].

Remark. As the ideal $\mathcal{I}$ is admissible, $\mathcal{I}_f \subseteq \mathcal{I}$. Hence every convergent sequence is $\mathcal{I}$-convergent.

For any ideal $\mathcal{I}$ there is a filter $\mathcal{F}(\mathcal{I})$ corresponding with $\mathcal{I}$:

$$\mathcal{F}(\mathcal{I}) = \big\{K \subseteq \mathbb{N} : (\mathbb{N} \setminus K) \in \mathcal{I}\big\}.$$

Lemma 2.4 ([5, Proposition 1.2])**.** *Let X be a normed linear space and $x = (x_n) \in w(X)$. If there is a set $K = \big\{n_1 < n_2 < \dots\big\} \in \mathcal{F}(\mathcal{I})$ such that x converges along K to $\xi \in X$ (*i.e.*, $\xi = \lim_{i\to\infty} x_{n_i}$), then x $\mathcal{I}$-converges to ξ.*

Lemma 2.5. *Let $D \in \mathcal{F}(\mathcal{I})$ and $M \subseteq \mathbb{N}$. If $M \notin \mathcal{I}$ then $M \cap D \notin \mathcal{I}$.*

Proof. Assume that $M \notin \mathcal{I}$ and $M \cap D \in \mathcal{I}$. Then $K = (N \setminus D) \cup (M \cap D) \in I$. But $M \subseteq K$ and so $M \in \mathcal{I}$. This is a contradiction to the assumption.

Therefore $M \cap D \notin \mathcal{I}$. □

Definition 2.6. A subspace E of space $w(X)$ is said to be *solid* if $x = (x_n) \in E$ and $\|y_n\| \le \|x_n\|$ for all $n \in \mathbb{N}$ implies $y = (y_n) \in E$.

Definition 2.7. If $A = (A_{nk})$ is an infinite matrix of operators $A_{nk} \in B(X,Y)$ ($B(X,Y)$ being the set of all bounded linear operators from X to Y) and $x = (x_n) \in w(X)$, then we say that x is *$\mathcal{I}$-summable by A to $\xi \in Y$* if

$$A_n x = \sum_{k=1}^{\infty} A_{nk} x_k$$

$\mathcal{I}$-converges to ξ in the norm of Y. Further x is said to be *bounded $\mathcal{I}$-summable by A to $\xi \in Y$* if and only if $(A_n x)$ is bounded and $\mathcal{I}$-converges to ξ in the norm of Y.

For a linear normed space X we will denote $\big(c(X)\big)^{\mathcal{I}}$, $\big(c_0(X)\big)^{\mathcal{I}}$, $\big(m(Y)\big)^{\mathcal{I}}$ and $\big(m_0(Y)\big)^{\mathcal{I}}$ the sets of *$\mathcal{I}$-convergent*, *$\mathcal{I}$-null*, *bounded $\mathcal{I}$-convergent* and *bounded $\mathcal{I}$-null* sequences of points from X, respectively. In case $X = \mathbb{C}$ we will write simply $c^{\mathcal{I}}$, $c_0^{\mathcal{I}}$, $m^{\mathcal{I}}$ and $m_0^{\mathcal{I}}$, respectively.

Definition 2.8. *The $\mathcal{I}$-convergence field of a matrix A* is defined by

$$\big(c(Y)\big)_A^{\mathcal{I}} = \Big\{x = (x_k) \in w(X) : (A_n x) \in \big(c(Y)\big)^{\mathcal{I}}\Big\}.$$

The bounded $\mathcal{I}$-convergence field of a matrix A is defined by

$$\big(m(Y)\big)_A^{\mathcal{I}} = \Big\{x = (x_k) \in w(X) : (A_n x) \in \big(m(Y)\big)^{\mathcal{I}}\Big\}$$

Since $c(Y) \subset \big(c(Y)\big)^{\mathcal{I}}$, so $\big(c(Y)\big)_A^{\mathcal{I}}$ is a natural generalization of $\big(c(Y)\big)_A$, *i.e.*, the convergence field of A (*cf.* [7]). We introduce the following notations:

$$\big(c_0(Y)\big)_A = \Big\{x = (x_k) \in w(X) : (A_n x) \in c_0(Y)\Big\},$$
$$\big(c_0(Y)\big)_A^{\mathcal{I}} = \Big\{x = (x_k) \in w(X) : (A_n x) \in \big(c_0(Y)\big)^{\mathcal{I}}\Big\}$$

and

$$\big(m_0(Y)\big)_A^{\mathcal{I}} = \Big\{x = (x_k) \in w(X) : (A_n x) \in \big(m_0(Y)\big)^{\mathcal{I}}\Big\}.$$

If $Y = \mathbb{C}$ we will omit the argument $\mathbb{C}$ and we will use the notations $c_A^{\mathcal{I}}$, $m_A^{\mathcal{I}}$, $(c_0)_A$, $(c_0)_A^{\mathcal{I}}$ and $(m_0)_A^{\mathcal{I}}$, respectively.

Throughout the paper we assume that $A_{nk} = \overline{\theta}$ if $n \neq k$, where $\overline{\theta}$ is the null operator from X to Y, *i.e.*, A is infinite diagonal matrix of operators. In this case $A_n x = A_{nn} x_n$ for $x = (x_n) \in w(X)$. We also restrict ourselves to the case $Y = \mathbb{C}$.

3. Main Results

The proof of the following proposition is obvious.

Proposition 3.1. *$(c_0)_A$, $c_A^{\mathcal{I}}$, $(c_0)_A^{\mathcal{I}}$, $m_A^{\mathcal{I}}$ and $(m_0)_A^{\mathcal{I}}$ are linear spaces.*

Recall that a non-trivial ideal $\mathcal{I}$ is said to be *maximal* if there does not exist any non-trivial ideal $\mathcal{J} \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset (*cf.* [5]).

Theorem 3.2. *If $m_A^{\mathcal{I}}$ is solid and $\mathcal{I}$ is not maximal, then*

$$\big\{n \in \mathbb{N} : A_{nn} = \overline{\theta}\big\} \notin \mathcal{I}. \tag{1}$$

Proof. Let us denote $B = \big\{n \in \mathbb{N} : A_{nn} = \overline{\theta}\big\}$ and assume that $B \in \mathcal{I}$.
Then for all $n \in D = \mathbb{N} \setminus B$ there exists $z_n \in X$ such that $A_{nn} z_n \neq 0$.
Define the sequences $x = (x_n),\ y = (y_n) \in w(X)$ as follows:

$$\begin{array}{lll} x_n = 0 & \text{for } n \in B, \\ x_n = \frac{z_n}{A_{nn} z_n} & \text{otherwise,} \end{array} \qquad \begin{array}{ll} y_n = 0 & \text{for } n \in B, \\ y_n = \frac{z_n}{A_{nn} z_n} & \text{for } n \in M \cap D, \\ y_n = 0 & \text{otherwise,} \end{array}$$

where $M \subset \mathbb{N}$ is such that

$$M \notin \mathcal{I} \text{ nor } (\mathbb{N} \setminus M) \notin \mathcal{I} \tag{2}$$

(such a set exists because $\mathcal{I}$ is not maximal).
One can see that

$$\begin{array}{ll} A_{nn} x_n = 0 & \text{for } n \in B, \\ A_{nn} x_n = 1 & \text{otherwise.} \end{array}$$

Hence $(A_{nn} x_n)$ is bounded and $\mathcal{I}\text{-}\lim_{n\to\infty} A_{nn} x_n = 1$, *i.e.*, $x \in m_A^{\mathcal{I}}$.
As $\|y_n\| \leq \|x_n\|$ for all $n \in \mathbb{N}$ and by the assumption $m_A^{\mathcal{I}}$ is solid, then $y \in m_A^{\mathcal{I}}$ too.
Let us enumerate the sequence $(A_{nn} y_n)$:

$$\begin{array}{ll} A_{nn} y_n = 0 & \text{for } n \in B, \\ A_{nn} y_n = 1 & \text{for } n \in M \cap D, \\ A_{nn} y_n = 0 & \text{otherwise.} \end{array}$$

Hence the $\mathcal{I}\text{-}\lim_{n\to\infty} A_{nn} y_n$ can be only 0 or 1.
Let $\mathcal{I}\text{-}\lim_{n\to\infty} A_{nn} y_n = 0$. Taking any $\varepsilon \in (0,1)$ we get $M \cap D = \big\{n \in \mathbb{N} : |A_{nn} y_n| \geq \varepsilon\big\} \in \mathcal{I}$. But this contradicts Lemma 2.5.
Let $\mathcal{I}\text{-}\lim_{n\to\infty} A_{nn} y_n = 1$. Taking any $\varepsilon \in (0,1)$ we get $\mathbb{N} \setminus (M \cap D) = \big\{n \in \mathbb{N} : |A_{nn} y_n - 1| \geq \varepsilon\big\} \in \mathcal{I}$. As $\mathbb{N} \setminus (M \cap D) = (\mathbb{N} \setminus M) \cup (\mathbb{N} \setminus D)$ then $(\mathbb{N} \setminus M) \in \mathcal{I}$ but this contradicts (2).

Hence $\mathcal{I}\text{-}\lim_{n\to\infty} A_{nn}y_n$ does not exist.
Therefore $m_A^{\mathcal{I}}$ is not solid and this completes the proof of the theorem. □

Throughout this section we will assume that ideal $\mathcal{I}$ is not maximal. By M we will denote a subset of $\mathbb{N}$ satisfying condition (2). The case of maximal ideal will be discussed later.

The following examples show that the condition (1) is not sufficient in general for the solidity of $m_A^{\mathcal{I}}$.

Example 3.3. Let $X = c_0$, the space of all complex sequences converging to zero with *sup-norm*. Define $A_{nn}s$ for $s = (s_n) \in c_0$ as follows:

$$\begin{aligned} A_{nn}s &= s_n \quad \text{for } n \in M, \\ A_{nn}s &= 0 \quad \text{otherwise.} \end{aligned}$$

The condition (1) is satisfied, but we show that $m_A^{\mathcal{I}}$ is not solid.

To prove this, construct the sequences $x = (x^{(n)})$ and $y = (y^{(n)})$, where $x^{(n)} = (x_{n1}, x_{n2}, \dots)$ and $y^{(n)} = (y_{n1}, y_{n2}, \dots)$ are in c_0, in the following way:

$$\begin{aligned} x_{n1} &= 1 \quad \text{for } n \in \mathbb{N}, & y_{nn} &= 1 \quad \text{for } n \in \mathbb{N}, \\ x_{nn} &= \tfrac{1}{n} \quad \text{for } n \in \mathbb{N}, & y_{nk} &= 0 \quad \text{for } k \neq n. \\ x_{nk} &= 0 \quad \text{for } k \neq n, k \geq 2, \end{aligned}$$

For all $n \in \mathbb{N}$ we have $\|y^{(n)}\| = \|x^{(n)}\|$.

One can easily verify that $\lim_{n\to\infty} A_{nn}x^{(n)} = 0$. Hence $\mathcal{I}\text{-}\lim_{n\to\infty} A_{nn}x^{(n)} = 0$. But $\mathcal{I}\text{-}\lim_{n\to\infty} A_{nn}y^{(n)}$ does not exist ($A_{nn}y^{(n)}$ equals 1 for $n \in M$ and 0 otherwise).

Therefore $y \notin m_A^{\mathcal{I}}$ and $m_A^{\mathcal{I}}$ is not solid.

Example 3.4. Let $X = l_1$, the space of all absolutely summable complex sequences. Define $A_{nn}s$ for $s = (s_n) \in l_1$ as follows:

$$\begin{aligned} A_{nn}s &= s_n \quad \text{for } n \in M, \\ A_{nn}s &= 0 \quad \text{otherwise.} \end{aligned}$$

The condition (1) is satisfied, but $m_A^{\mathcal{I}}$ is not solid.

Define the sequences $x = (x^{(n)})$ and $y = (y^{(n)})$ in $w(l_1)$ as follows: For all $n \in \mathbb{N}$ set $x^{(n)} = (x_{n1}, x_{n2}, \dots)$ and $y^{(n)} = (y_{n1}, y_{n2}, \dots)$, where

$$\begin{aligned} x_{n1} &= 1, & y_{nn} &= \tfrac{(-1)^n}{2}, \\ x_{nn} &= \tfrac{1}{n}, & y_{nk} &= 0 \quad \text{for } k \neq n. \\ x_{nk} &= 0 \quad \text{for } k \neq n, k \geq 2, \end{aligned}$$

We have $\|y^{(n)}\| = \sum_{k=1}^{\infty} |y_{nk}| = \frac{1}{2} \leq 1 + \frac{1}{n} = \sum_{k=1}^{\infty} |x_{nk}| = \|x^{(n)}\|$.

Because $\lim_{n\to\infty} A_{nn}x^{(n)} = 0$ and $\mathcal{I}\text{-}\lim_{n\to\infty} A_{nn}y^{(n)}$ does not exist, $m_A^{\mathcal{I}}$ is not solid.

The proof of the following result is a routine work in view of Theorem 3.2.

Theorem 3.5. *If $c_A^{\mathcal{I}}$ is solid then the condition* (1) *holds.*

Proposition 3.6. *Let $X = c_0$. Then $c_A^{\mathcal{I}}$ is solid if and only if*

$$\left\{ n \in \mathbb{N} : A_{nn} = \overline{\theta} \right\} \in \mathcal{F}(\mathcal{I}). \tag{3}$$

Proof. Let us denote $B = \{n \in \mathbb{N} : A_{nn} = \overline{\theta}\}$.

If $B \in \mathcal{F}(\mathcal{I})$ then for all $x = (x^{(n)}) \in w(c_0)$ we have: $\mathcal{I}\text{-}\lim_{n\to\infty} A_{nn}x^{(n)} = 0$ (*see* Lemma 2.4). Therefore $c_A^{\mathcal{I}} = w(c_0)$ and $w(c_0)$ is solid. So the sufficiency of condition (3) is proved.

Now assume that $c_A^{\mathcal{I}}$ is solid and $B \notin \mathcal{F}(\mathcal{I})$, *i.e.*, $D = \mathbb{N} \setminus B = \{n \in \mathbb{N} : A_{nn} \neq \overline{\theta}\} \notin \mathcal{I}$.

For every $n \in \mathbb{N}$, A_{nn} is a bounded linear functional. Hence it has the form

$$A_{nn}s = \sum_{k=1}^{\infty} a_{nk}s_k \qquad \text{for all } s = (s_k) \in c_0,$$

where

$$\sum_{k=1}^{\infty} |a_{nk}| < \infty. \tag{4}$$

Let $\{n_1 < n_2 < \dots\}$ be the enumeration of D. Since $A_{n_in_i} \neq \overline{\theta}$, there is $k_i \in \mathbb{N}$ such that $a_{n_ik_i} \neq 0$ for $i = 1, 2, \dots$. Condition (4) implies that there exists an $r_i > k_i$ such that

$$|a_{n_ir_i}| < \frac{1}{i}|a_{n_ik_i}|.$$

Define the sequences $x = (x^{(n)})$ and $y = (y^{(n)})$, where $x^{(n)} = (x_{n1}, x_{n2}, \dots)$ and $y^{(n)} = (y_{n1}, y_{n2}, \dots)$ are in c_0 for all $n \in \mathbb{N}$, as follows:

$$\begin{array}{lll} x_{nk} = \frac{1}{ia_{n_ik_i}} & \text{for } n = n_i, k = k_i, \\ x_{nk} = \frac{1}{a_{n_ik_i}} & \text{for } n = n_i, k = r_i, \\ x_{nk} = 0 & \text{otherwise,} \end{array} \qquad \begin{array}{ll} y_{nk} = \frac{1}{a_{n_ik_i}} & \text{for } n = n_i, k = k_i, \\ y_{nk} = 0 & \text{otherwise.} \end{array}$$

It is clear that $\|x^{(n)}\| = \|y^{(n)}\|$ for all $n \in \mathbb{N}$.

Let us look at the sequences $(A_{nn}x^{(n)})$ and $(A_{nn}y^{(n)})$: If $n \in B$ then $A_{nn}x^{(n)} = A_{nn}y^{(n)} = 0$.

Let $n = n_i$. Then

$$\begin{array}{l} |A_{n_in_i}x^{(n_i)}| = |\sum_{k=1}^{\infty} a_{n_ik}x_{n_ik}| = |a_{n_ik_i}x_{n_ik_i} + a_{n_ir_i}x_{n_ir_i}| \leq \frac{2}{i}, \\ |A_{n_in_i}y^{(n_i)}| = |\sum_{k=1}^{\infty} a_{n_ik}y_{n_ik}| = |a_{n_ik_i}y_{n_ik_i}| = 1. \end{array}$$

It follows that $\mathcal{I}\text{-}\lim_{n\to\infty} A_{nn}x^{(n)} = 0$. But $\mathcal{I}\text{-}\lim_{n\to\infty} A_{nn}y^{(n)}$ does not exist, since $A_{nn}y^{(n)} = 0$ for $n \in B$ and $A_{nn}y^{(n)} = 1$ for $n \in D$ with $B \notin \mathcal{I}$ (by Theorem 3.5) and $D \notin \mathcal{I}$ (by assumption).

Thus $c_A^{\mathcal{I}}$ is not solid.

□

The following example demonstrates that $c_A^{\mathcal{I}}$ in the previous proposition cannot be replaced by $m_A^{\mathcal{I}}$.

Example 3.7. Let $X = c_0$. Let $D \in \mathcal{I}$ be an infinite set. (If such a D does not exist, then $\mathcal{I} = \mathcal{I}_f$ and in this case $c_A^{\mathcal{I}} = m_A^{\mathcal{I}}$.) Define $A_{nn}s$ for $s = (s_n) \in c_0$ as follows:

$$\begin{array}{ll} A_{nn}s = ns_1 + ns_2, & \text{for } n \in D, \\ A_{nn}s = 0, & \text{otherwise.} \end{array}$$

Then the condition (3) is satisfied, but $m_A^{\mathcal{I}}$ is not solid.

Take the sequences $x = (x^{(n)})$ and $y = (y^{(n)})$ in $w(c_0)$, where

$$x^{(n)} = (1, -1, 0, 0, \dots),$$
$$y^{(n)} = (1, 0, 0, \dots)$$

for all $n \in \mathbb{N}$. Then $\|y^{(n)}\| = \|x^{(n)}\|$ and $x \in m_A^{\mathcal{I}}$, but $(A_{nn}y^{(n)})$ is unbounded, thus $y \notin m_A^{\mathcal{I}}$.

Proposition 3.8. *Let $X = \mathbb{C}$. Then $m_A^{\mathcal{I}}$ ($c_A^{\mathcal{I}}$) is solid if and only if the condition* (1) *holds.*

Proof. We prove the statement of the proposition for $m_A^{\mathcal{I}}$ only. The proof for $c_A^{\mathcal{I}}$ runs in a similar way.

If $m_A^{\mathcal{I}}$ is solid, then the condition (1) holds by Theorem 3.2. Hence the necessity of (1) is proved.

Let us prove the sufficiency of (1).

Assume that (1) holds. Take $x = (x_n) \in m_A^{\mathcal{I}}$ and let $y = (y_n) \in w$ be such that $\|y_n\| \leq \|x_n\|$ for all $n \in \mathbb{N}$. We claim that $y \in m_A^{\mathcal{I}}$.

Every operator A_{nn} has the form $A_{nn}s = a_n s$ for $s \in \mathbb{C}$, where $a_n \in \mathbb{C}$.

Let $\xi = \mathcal{I}\text{-}\lim_{n\to\infty} a_n x_n$. Then for all $\varepsilon > 0$ the following holds:

$$\big\{n \in \mathbb{N} : |a_n x_n - \xi| \geq \varepsilon\big\} \in \mathcal{I}.$$

Since (1) holds, $\xi = 0$.

By assumption for all $n \in \mathbb{N}$ we have:

$$|a_n y_n| = |a_n||y_n| \leq |a_n||x_n| = |a_n x_n|.$$

This means that $(a_n y_n)$ is bounded ($x \in m_A^{\mathcal{I}}$) and

$$\big\{n \in \mathbb{N} : |a_n y_n| \geq \varepsilon\big\} \subseteq \big\{n \in \mathbb{N} : |a_n x_n| \geq \varepsilon\big\}$$

Thus $\mathcal{I}\text{-}\lim_{n\to\infty} a_n y_n = 0$ and $y \in m_A^{\mathcal{I}}$. □

Proposition 3.9. *Let $X = \mathbb{C}$. Then the spaces $(c_0)_A$, $(c_0)_A^{\mathcal{I}}$ and $(m_0)_A^{\mathcal{I}}$ are solid for any infinite diagonal matrix A.*

Proof. Let $A = (a_{nk})$ be a diagonal matrix. Further, let $x = (x_n) \in (m_0)_A^{\mathcal{I}}$ and $y = (y_n) \in w$ be such that $|y_n| \leq |x_n|$ for all $n \in \mathbb{N}$. Then we have

$$|a_{nn} y_n| \leq |a_{nn} x_n| \qquad \text{for all } n \in \mathbb{N}.$$

As $\mathcal{I}\text{-}\lim_{n\to\infty} a_{nn} x_n = 0$ then also $\mathcal{I}\text{-}\lim_{n\to\infty} a_{nn} y_n = 0$. The boundedness of $(a_{nn} y_n)$ is clear. Thus $(y_n) \in (m_0)_A^{\mathcal{I}}$.

Similarly one can show that $(c_0)_A$ and $(c_0)_A^{\mathcal{I}}$ are solid too. □

4. Case of maximal ideal

Definition 4.1. Let X be a normed linear space. We say that $x = (x_n) \in w(X)$ is *$\mathcal{I}$-bounded* if there is a number $M > 0$ such that the following holds:

$$\big\{n \in \mathbb{N} : \|x_n\| > M\big\} \in \mathcal{I}.$$

Remark. Every bounded sequence is $\mathcal{I}$-bounded.

Lemma 4.2. *Let $\mathcal{I}$ be maximal. Then every $\mathcal{I}$-bounded sequence $x = (x_n)$ of complex numbers is $\mathcal{I}$-convergent.*

Proof. Let $x = (x_n)$ be $\mathcal{I}$-bounded sequence of complex numbers. Then there is a closed square $U^{(1)}$ in complex plane such that the set of indices of elements of x lying outside of $U^{(1)}$ belongs to $\mathcal{I}$, *i.e.*,

$$\{n \in \mathbb{N} : x_n \in U^{(1)}\} \in \mathcal{F}(\mathcal{I}).$$

Divide $U^{(1)}$ into four closed squares $U_1^{(1)}, U_2^{(1)}, U_3^{(1)}, U_4^{(1)}$ with

$$\operatorname{diam} U_i^{(1)} = \frac{1}{2} \operatorname{diam} U^{(1)} \qquad (i = 1, 2, 3, 4).$$

The following holds at least for one of these squares:

$$\{n \in \mathbb{N} : x_n \in U_i^{(1)}\} \in \mathcal{F}(\mathcal{I}).$$

Denote this square $U^{(2)}$.

In this way, by induction, we construct a sequence

$$U^{(1)} \supset U^{(2)} \supset \cdots \supset U^{(n)} \supset \ldots$$

of closed squares with the properties

$$\{k \in \mathbb{N} : x_k \in U^{(n)}\} \in \mathcal{F}(\mathcal{I}) \qquad \text{for all } n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} \operatorname{diam} U^{(n)} = 0.$$

By Cantor's Theorem there exists (the only)

$$\xi \in \bigcap_{n\in\mathbb{N}} U^{(n)}.$$

We claim that $\xi = \mathcal{I}\text{-}\lim_{n\to\infty} x_n$.

Let $\varepsilon > 0$. Take a sphere $B(\xi, \varepsilon) = \{z \in \mathbb{C} : |z - \xi| < \varepsilon\}$. Then for all sufficiently large $m \in \mathbb{N}$ we have:

$$U^{(m)} \subseteq B(\xi, \varepsilon).$$

Fix such an m. Then

$$\{n \in \mathbb{N} : x_n \in U^{(m)}\} \subseteq \{n \in \mathbb{N} : x_n \in B(\xi, \varepsilon)\}.$$

As the left side belongs to $\mathcal{F}(\mathcal{I})$ so does the right side. Thus

$$\{n \in \mathbb{N} : |x_n - \xi| \geq \varepsilon\} \in \mathcal{I}.$$

Hence $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi$. □

Proposition 4.3. *Let $X = \mathbb{C}$ and $\mathcal{I}$ be maximal. Then $m_A^{\mathcal{I}}$ ($c_A^{\mathcal{I}}$) is solid for every infinite diagonal matrix A.*

Proof. Let $A = (a_{ij})$ be a diagonal matrix (*i.e.*, $a_{ij} = 0$ for $i \neq j$) and $x = (x_n) \in m_A^{\mathcal{I}}$. Then $(a_{nn}x_n)$ is bounded $\mathcal{I}$-convergent sequence of complex numbers.

Let $y = (y_n) \in w$ be such that

$$|y_n| \leq |x_n| \qquad \text{for all } n \in \mathbb{N}.$$

Hence

$$|a_{nn}y_n| \leq |a_{nn}x_n| \qquad \text{for all } n \in \mathbb{N}.$$

Thus $(a_{nn}y_n)$ is bounded and so the solidity of $m_A^{\mathcal{I}}$ follows easily from Lemma 4.2.

The proof for $c_A^{\mathcal{I}}$ uses the fact that every $\mathcal{I}$-convergent sequence is $\mathcal{I}$-bounded.

□

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