Range of density measures

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Abstract. We investigate some properties of density measures – finitely additive measures on the set of natural numbers $\mathbb{N}$ extending asymptotic density. We introduce a class of density measures, which is defined using cluster points of the sequence $\left(\frac{A(n)}{n}\right)$ as well as cluster points of some other similar sequences.

We obtain range of possible values of density measures for any subset of $\mathbb{N}$. Our description of this range simplifies the description of Bhashkara Rao and Bhashkara Rao [4] for general finitely additive measures. Also the values which can be attained by the measures defined in the first part of the paper are studied.

Introduction

We are interested in finitely additive measures defined on the algebra $\mathcal{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$. By a measure we mean a function $\mu : \mathcal{P}(\mathbb{N}) \to [0,1]$ satisfying the following properties:

(a) $\mu(\mathbb{N}) = 1$;

(b) $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subseteq \mathbb{N}$.

The *asymptotic density* d defined by $d(A) = \lim\limits_{n\to\infty} \frac{A(n)}{n}$, where $A(n) = \big|A \cap [1,n]\big|$, is a classical tool for measuring the size of subsets of $\mathbb{N}$. But unfortunately, it is not defined on all subsets of $\mathbb{N}$. Moreover, it is well known that the collection of sets having asymptotic density (the domain of d) does not form an algebra of sets. Let us denote this collection $\mathcal{D}$.

Clearly, $\mathbb{N} \in \mathcal{D}$ and $d(\mathbb{N}) = 1$. If $A, B \in \mathcal{D}$ and $A \cap B = \emptyset$, then $d(A \cup B) = d(A) + d(B)$. Hence d possesses both properties (a) and (b) above and it is known that it is possible to extend d to a measure.

We will study these extensions, i.e., the measures satisfying:

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(c) $\mu|_{\mathcal{D}} = d$.

This kind of measure will be called a *density measure* (in accordance with [26]).

Existence of density measures was shown already by S. Banach. In functional analysis it is usually proved using Hahn-Banach theorem (see e.g. [3, p.141,§3]). We will use a different approach for constructing density measures, using ultrafilters (see e.g. [2, Theorem 8.33], [19, p.207]). Also the general theory of extensions of a partial finitely additive measure to a measure, described in detail by Bhashkara Rao and Bhashkara Rao in [4], is a very convenient tool in this setting.

Dorothy Maharam [22] pioneered the research of the density measures on integers. This field was further studied by Blass, Frankiewicz, Plebanek and Ryll–Nardzewski in [5], van Douwen in [30] or Šalát and Tijdeman in [26]. Recently the density measures and related concept of Lévy group have been employed in the theory of social choice [8], [9], [21], [29].

Let us note that at least some form of axiom of choice is needed in the construction of finitely additive measures on $\mathbb{N}$, since there exists a model of ZF constructed by Pincus and Solovay [23] in which there are no nonprincipal finitely additive measures on $\mathbb{N}$, see also [18]. (It was mistakenly stated in [14] that Buck's measure [7] yields an effective construction of a density measure.)

1 Expressions of density measures

We start by describing the construction of density measures via ultrafilters.

We first recall the notion of *limit along a filter* (see [2, p.122, Definition 8.23], [19, p.206, Definition 2.7]). If $\mathcal{F}$ is a filter on $\mathbb{N}$ and (x_n) is a real sequence then we say that $\mathcal{F}\text{-}\lim x_n = L$ if L is a real number with the property

$$\{n; |x_n - L| > \varepsilon\} \in \mathcal{F}$$

for each $\varepsilon > 0$.

We recall here some basic (and easy to show) properties of the $\mathcal{F}$-limit, which will be needed later.

Lemma 1. *Let $\mathcal{F}$ be a free filter on $\mathbb{N}$ and (x_n) be a real sequence.*

(i) *If* $\lim\limits_{n\to\infty} x_n = L$ *then* $\mathcal{F}\text{-}\lim x_n = L$.

(ii) *If* $\mathcal{F}\text{-}\lim x_n$ *exists, then* $\liminf x_n \leq \mathcal{F}\text{-}\lim x_n \leq \limsup x_n$.

(iii) *The $\mathcal{F}$-limits are unique.*

(iv) $\mathcal{F}\text{-}\lim(ax_n + by_n) = a\,\mathcal{F}\text{-}\lim x_n + b\,\mathcal{F}\text{-}\lim y_n$ *(provided the $\mathcal{F}$-limits of (x_n) and (y_n) exist).*

(v) $\mathcal{F}\text{-}\lim(x_n.y_n) = \mathcal{F}\text{-}\lim x_n.\mathcal{F}\text{-}\lim y_n$ *(provided the $\mathcal{F}$-limits of (x_n) and (y_n) exist).*

(vi) *For every cluster point c of the sequence (x_n) there exists a free filter $\mathcal{F}$ such that $\mathcal{F}\text{-}\lim x_n = c$. On the other hand, if $\mathcal{F}\text{-}\lim x_n$ exists, it is a cluster point of the sequence (x_n).*

(vii) $\lim_{n\to\infty} x_n = L$ *if and only if* $\mathcal{F}\text{-}\lim x_n = L$ *for every free ultrafilter* $\mathcal{F}$.

(viii) *If* (x_n) *is bounded an* $\mathcal{F}$ *is an ultrafilter, then* $\mathcal{F}\text{-}\lim x_n$ *exists.*

Using the above properties of $\mathcal{F}$-limit one can show that for any free ultrafilter $\mathcal{F}$ on $\mathbb{N}$ a density measure $\mu_{\mathcal{F}}$ can be defined by

$$\mu_{\mathcal{F}}(A) = \mathcal{F}\text{-}\lim \frac{A(n)}{n}.$$

(We refer again to [2, Theorem 8.33], [19, p.207] for the proof of this claim.)

A short notice of Lauwers in [21] claims:

Every density measure can be expressed in the form

$$\mu_\varphi(A) = \int_{\beta\mathbb{N}^*} \mathcal{F}\text{-}\lim \frac{A(n)}{n}\, \mathrm{d}\varphi(\mathcal{F}), \qquad A \subseteq \mathbb{N} \tag{1.1}$$

for some probability Borel measure φ *on the set of all free ultrafilters* $\beta\mathbb{N}^*$.
But unfortunately our next considerations show that this result is not correct.

2 Density measures from α-densities

In this section we will consider another class of density measures. In order to define them we need to recall the definition of α-densities.

For $\alpha \geq -1$ and $A \subseteq \mathbb{N}$ we denote $A_\alpha(n) = \sum_{k=1}^{n} \chi_A(k)k^\alpha$ and by $\mathcal{D}_\alpha$ the set of all sets $A \subseteq \mathbb{N}$ such that the sequence $\left(\frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}\right)$ has a limit. The limit of this sequence we denote $d_\alpha(A)$ and we will call it the *α-density* of the set A, i.e., $d_\alpha(A) = \lim_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}$. Hence for $\alpha = 0$ we have the asymptotic density and for $\alpha = -1$ the logarithmic density.

As usual, by $\underline{d}$ and $\overline{d}$ we will denote the *lower* and the *upper asymptotic density*, respectively, i.e., $\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n}$ and $\overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}$. Similarly, we will call the functions $\underline{d_\alpha}(A) = \liminf_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}$ and $\overline{d_\alpha}(A) = \limsup_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}$ the *lower* and the *upper α-density*.

The following theorem is a consequence of the result of Fuchs and Giuliano Antonini in [11].

Theorem 1. *Let* $\alpha > -1$ *and* $f : \mathbb{N} \to \mathbb{R}$ *be a bounded arithmetic function. If*

$$\lim_{n\to\infty} \frac{1}{\mathbb{N}_\alpha(n)} \sum_{k=1}^{n} f(k)k^\alpha = L,$$

then

$$\lim_{n\to\infty} \frac{1}{\mathbb{N}_\beta(n)} \sum_{k=1}^{n} f(k)k^\beta = L$$

for any $\beta \geq -1$.

Replacing the function f by the characteristic function of a set A we get

Corollary 1. *If $A \in \mathcal{D}_\alpha$ for some $\alpha > -1$, i.e., the α-density $d_\alpha(A) = \lim\limits_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}$ of a set A exists, then $A \in \mathcal{D}_\beta$ for all $\beta \geq -1$, and $d_\alpha(A) = d_\beta(A)$.*

Corollary 2. *For all $\alpha > -1$ we have $\mathcal{D}_\alpha = \mathcal{D}$ and $d_\alpha = d$.*

This means that by replacing the sequence $\left(\frac{A(n)}{n}\right)$ in (1.1) by the sequence $\left(\frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}\right)$ for some $\alpha > -1$ we get a density measure. By well-known inequality

$$\underline{d_{-1}} \leq \underline{d} \leq \overline{d} \leq \overline{d_{-1}}$$

(see [17, p.241,Lemma V.2.1], [28, p.272]) we get $\mathcal{D} \subseteq \mathcal{D}_{-1}$ and $d_{-1}|_{\mathcal{D}} = d$. Therefore d_α is an extension of d for $\alpha = -1$, too.

In particular, if we fix some $\alpha \geq -1$ and some free ultrafilter $\mathcal{F}$, then the mapping $A \mapsto \mathcal{F}\text{-lim} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}$ defines a density measure. Let us denote this density measure by $\mu_\alpha^{\mathcal{F}}$.

The following lemma can be useful for evaluating α-densities.

Lemma 2. *For all $\alpha > -1$ we have*

$$\lim_{n\to\infty} \frac{n^{\alpha+1}}{\mathbb{N}_\alpha(n)} = \alpha + 1,$$

and for $\alpha = -1$

$$\lim_{n\to\infty} \frac{\ln(n)}{\mathbb{N}_{-1}(n)} = 1.$$

The routine proof can be done for example by interpreting the sums appearing in the definition of $\mathbb{N}_\alpha(n)$ as the lower and upper Riemann sums for integral of the function x^α or by using Stolz theorem.

Now we will show that for every $\alpha > 0$, there is a free ultrafilter $\mathcal{F}$ such that $\mu_\alpha^{\mathcal{F}}$ is different from all density measures μ_φ expressible by (1.1).

Since the value of $\mathcal{F}\text{-lim} \frac{A(n)}{n}$ is a cluster point of the sequence $\frac{A(n)}{n}$, we see that $\underline{d}(A) \leq \mathcal{F}\text{-lim} \frac{A(n)}{n} \leq \overline{d}(A)$ for all free ultrafilters $\mathcal{F}$ and all $A \subseteq \mathbb{N}$, and consequently $\underline{d} \leq \mu_\varphi \leq \overline{d}$ for every probability Borel measure φ on $\beta\mathbb{N}^*$.

Finally we are ready to present a counterexample to the Lauwers' assertion: Let $A = \bigcup\limits_{k=0}^{\infty} \left(2^{2k}, 2^{2k+1}\right] \cap \mathbb{N}$. Similarly as in Lemma 2 one can show that:

$$\underline{d_\alpha}(A) = \lim_{k\to\infty} \frac{A_\alpha(2^{2k})}{\mathbb{N}_\alpha(2^{2k})} = \lim_{k\to\infty} \frac{\sum_{i=0}^{k-1} \int_{2^{2i}}^{2^{2i+1}} x^\alpha \,\mathrm{d}x}{\frac{(2^{2k})^{\alpha+1}}{\alpha+1}} = \frac{2^{\alpha+1}-1}{2^{2(\alpha+1)}-1} = \frac{1}{2^{\alpha+1}+1},$$

$$\overline{d_\alpha}(A) = \lim_{k\to\infty} \frac{A_\alpha(2^{2k+1})}{\mathbb{N}_\alpha(2^{2k+1})} = \lim_{k\to\infty} \frac{\sum_{i=0}^{k} \int_{2^{2i}}^{2^{2i+1}} x^\alpha \,\mathrm{d}x}{\frac{(2^{2k+1})^{\alpha+1}}{\alpha+1}} = \frac{2^{\alpha+1}(2^{\alpha+1}-1)}{2^{2(\alpha+1)}-1}$$
$$= \frac{2^{\alpha+1}}{2^{\alpha+1}+1}$$

for all $\alpha > -1$. Hence $\underline{d_\beta}(A) < \underline{d_\alpha}(A) < \overline{d_\alpha}(A) < \overline{d_\beta}(A)$, if $-1 < \alpha < \beta$.

Now, taking any free ultrafilter $\mathcal{F}$ containing the set $\{2^{2k}; k = 0, 1, \dots\}$ we get:

$$\mu_\alpha^{\mathcal{F}}(A) = \mathcal{F}\text{-lim} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)} = \underline{d_\alpha}(A).$$

This shows that the measure $\mu_\alpha^{\mathcal{F}}$ cannot be of the form (1.1), if $\alpha > 0$.

Apart from providing a counterexample to (1.1), this answers also one part of [30, Question 7A.1]. Van Douwen asks, whether $\mu(A) \le \overline{d}(A)$ for every density measure. The above procedure yields a measure $\mu_\alpha^{\mathcal{F}}$ with $\mu_\alpha^{\mathcal{F}}(A) = \overline{d_\alpha}(A) > \overline{d}(A)$ (for $\alpha > 0$ and appropriate choice of the free filter $\mathcal{F}$). A different example, based on results of Blümlinger [6], was presented in [27].

Our previous observations lead to a more general class of density measures than the one defined by Lauwers.

If a measure μ can be expressed in the form

$$\mu(A) = \int_\Omega \mu_\alpha^{\mathcal{F}}(A)\, \mathrm{d}\psi(\mathcal{F}, \alpha), \qquad A \subseteq \mathbb{N} \tag{2.1}$$

for some probability Borel measure ψ on the set $\Omega = \beta\mathbb{N}^* \times [-1, \infty)$, then μ is a density measure.

To be precise, we should check the existence of the integral in (2.1). As the function $f(\mathcal{F}, \alpha) = \mu_\alpha^{\mathcal{F}}(A) = \mathcal{F}\text{-lim} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}$ is bounded, it suffices to show that it is ψ-measurable for every Borel measure ψ. By Johnson [20] a sufficient condition for $f(\mathcal{F}, \alpha)$ to be measurable is its separate continuity, i.e., continuity in $\mathcal{F}$ for any fixed α and continuity in α for any fixed $\mathcal{F}$.

The continuity in $\mathcal{F}$ follows immediately from the general theory of the Stone-Čech compactification of a topological space (see e.g. [12] or [31]).

For $\alpha > -1$, the continuity in α follows from the estimations of Giuliano Antonini, Grekos and Mišík [13]:

$$\limsup_{n\to\infty} \left| \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)} - \frac{A_{\alpha+\delta}(n)}{\mathbb{N}_{\alpha+\delta}(n)} \right| < \frac{2\delta}{\alpha+1}$$
$$\limsup_{n\to\infty} \left| \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)} - \frac{A_{\alpha-\delta}(n)}{\mathbb{N}_{\alpha-\delta}(n)} \right| < \frac{2\delta}{\alpha-\delta+1}$$

for $0 < \delta < \alpha + 1$.

It is proved in [13] that there exists a set A such that the function $\underline{d_\alpha}(A)$ is discontinuous at $\alpha = -1$. Thus our function $f(\mathcal{F}, \alpha)$ cannot be continuous at $\alpha = -1$ for all filters $\mathcal{F} \in \mathbb{N}^*$. Hence we get the separate continuity on $\beta\mathbb{N}^* \times (-1, \infty)$, only.

So f is ψ-measurable on $\beta\mathbb{N}^* \times (-1, \infty)$ and on the measurable set $\beta\mathbb{N}^* \times \{-1\}$ it is continuous, and thus Borel measurable. It follows that f is measurable on Ω.

3 Values of density measures

Assume that μ is a density measure. Let $A \subseteq \mathbb{N}$. The question is: *What are the possible values of* $\mu(A)$. Or: *Which values can be attained by all density measures for a fixed set A?* This question was proposed by Mark Fey in [10].

It is clear that if $A \in \mathcal{D}$, then $\mu(A) = d(A)$ for all density measures μ. But if it is not the case, there are more possibilities for the value of $\mu(A)$. The next paragraphs answer the above question.

The first estimation of $\mu(A)$ can be made using monotonicity of a measure.

If $B \subseteq A$ and $B \in \mathcal{D}$, then $d(B) = \mu(B) \leq \mu(A)$. Hence $\sup\{d(B);\ B \subseteq A,\ B \in \mathcal{D}\} \leq \mu(A)$. Similarly, $\inf\{d(C);\ C \supseteq A,\ C \in \mathcal{D}\} \geq \mu(A)$. Let us denote

$$\underline{\underline{d}}(A) = \sup\{d(B);\ B \subseteq A,\ B \in \mathcal{D}\},$$
$$\overline{\overline{d}}(A) = \inf\{d(C);\ C \supseteq A,\ C \in \mathcal{D}\}.$$

Thus we get

Theorem 2. *For every set $A \subseteq \mathbb{N}$ and all density measures μ we have:*

$$\underline{\underline{d}}(A) \leq \mu(A) \leq \overline{\overline{d}}(A). \tag{3.1}$$

Later on we will show that this estimation is the best possible.

Let us take

$$d_*(A) = \sup \frac{\sum_{i=1}^{p} d(A_i) - \sum_{j=1}^{q} d(B_j)}{k}.$$

The supremum is taken over all finite collections $A_1, A_2, \dots, A_p,\ B_1, B_2, \dots B_q$ of sets in $\mathcal{D}$ and positive integers k such that

$$k\chi_A + \sum_{j=1}^{q} \chi_{B_j} \geq \sum_{i=1}^{p} \chi_{A_i}.$$

Similarly,

$$d^*(A) = \inf \frac{\sum_{i=1}^{p} d(A_i) - \sum_{j=1}^{q} d(B_j)}{k}.$$

The infimum is taken over all finite collections $A_1, A_2, \dots, A_p,\ B_1, B_2, \dots B_q \in \mathcal{D}$ and positive integers k such that

$$k\chi_A + \sum_{j=1}^{q} \chi_{B_j} \leq \sum_{i=1}^{p} \chi_{A_i}$$

It is clear that

$$\underline{\underline{d}}(A) \leq d_*(A) \leq d^*(A) \leq \overline{\overline{d}}(A). \tag{3.2}$$

By Bhashkara Rao [4, Theorem 3.2.9] for every set $A \subseteq \mathbb{N}$ and any value $x \in [d_*(A), d^*(A)]$ there is a density measure μ such that $\mu(A) = x$. Moreover, if μ is a density measure, then $\mu(A) \in [d_*(A), d^*(A)]$.

The definition of d_* and d^* (the range of density measures) is rather complicated. The original result in [4] was formulated for more general situation of extending arbitrary partial measures. (Roughly said, by a partial measure we mean a restriction of a measure to some class of subsets of $\mathbb{N}$. For more details we refer the reader to [4, Section 3.2].) In our case, we work only with the asymptotic density and our aim is to prove the simplification of this result. This simplification is contained in the following theorem and its corollary.

Theorem 3. *For every $A \subseteq \mathbb{N}$ the following is true:*

$$\underline{\underline{d}}(A) = d_*(A) \qquad \text{and} \qquad \overline{\overline{d}}(A) = d^*(A). \tag{3.3}$$

Corollary 3. *Let $A \subseteq \mathbb{N}$. There exists a density measure μ such that $\mu(A) = x$ if and only if $x \in [\underline{\underline{d}}(A), \overline{\overline{d}}(A)]$.*

Let us note that if a partial measure m is defined on an algebra of sets then by a result due to Łoś and Marczewski [4, Proposition 3.3.1] $\underline{\underline{m}} = m_*$ and $\overline{\overline{m}} = m^*$ holds for this measure. This result cannot be used here, since $\mathcal{D}$ is not closed under intersections and unions. (In fact, the smallest algebra containing $\mathcal{D}$ is the whole powerset $\mathcal{P}(\mathbb{N})$.)

As we show in Remark 1, Theorem 3 and Corollary 3 could be deduced from results of Pólya [24] using some functional analytic considerations. However, we still find our proof of interest, since it is relatively elementary and it is an interesting application of known results on density sets obtained by Grekos and Volkmann [16].

Before we prove Theorem 3 we will describe some basic properties of $\underline{\underline{d}}$ and $\overline{\overline{d}}$. The following lemma is crucial for proving some of them. Let us note that the proof was inspired by the proof of [26, Lemma 1].

Lemma 3. *If $A, B \in \mathcal{D}$, $d(A) < d(B)$, then there exists $D \in \mathcal{D}$ such that $A \cap B \subseteq D \subseteq B$ and $d(D) = d(A)$.*

Proof. Put $C = A \cap B$, $A' = A \setminus C$, $B' = B \setminus C$. We have $\lim\limits_{n\to\infty} \frac{C(n)+B'(n)}{n} = d(B)$ and $\lim\limits_{n\to\infty} \frac{C(n)+A'(n)}{n} = d(A)$, hence

$$L = \lim_{n\to\infty} \frac{B'(n) - A'(n)}{n} = d(B) - d(A) > 0.$$

We shall construct a subset $D' \subset B'$ such that $\lim\limits_{n\to\infty} \frac{D'(n)-A'(n)}{n} = 0$. Then for $D = C \cup D'$ we have $d(D) = \lim\limits_{n\to\infty} \frac{C(n)+D'(n)}{n} = \lim\limits_{n\to\infty} \frac{C(n)+A'(n)}{n} + \lim\limits_{n\to\infty} \frac{D'(n)-A'(n)}{n} = d(A)$, so D is the desired subset of B.

The subset D' is defined by induction. If $n \notin B'$, then $n \notin D'$. If $n \in B'$ and $D'(n-1) + 1 > A'(n)$, then $n \notin D'$. If $n \in B'$ and $D'(n-1) + 1 \leq A'(n)$, then $n \in D'$. It is obvious that $D'(n) \leq A'(n)$.

Let us note that if $m \in B'$ but $m \notin D'$ (the second case), then $D'(m) + 1 > A'(m) \geq D'(m)$, hence $D'(m) = A'(m)$. If $n \in \mathbb{N}$ and m is the largest number such that $m \leq n$, $m \notin D'$ and $m \in B'$, then for every k, $m < k \leq n$, we have $D'(k) - D'(m) = B'(k) - B'(m)$ (since all members of B' in the interval $(m, n]$ belong to D'). This implies $B'(k) - B'(m) \leq A'(k) - A'(m)$.

We denote the largest number $m \leq n$ for which the second case occurs by $m(n) = m$. The set $\{m(n); n \in \mathbb{N}\}$ of all such numbers is unbounded. Otherwise, assume that m is the maximal element of this set. Then we get $d(B) = \lim\limits_{n\to\infty} \frac{C(n)+B'(n)}{n} \leq \lim\limits_{n\to\infty} \frac{C(n)+A'(n)+B'(m)-A'(m)}{n} = d(A)$, a contradiction.

Now let $\varepsilon > 0$ and N_0 be such that for $k \geq N_0$ the inequality $\left|\frac{B'(n)-A'(n)}{n} - L\right| \leq \varepsilon$ holds. Since the set $\{m(n);\, n \in \mathbb{N}\}$ is unbounded, we can choose n large enough to assure that $n \geq m(n) \geq N_0$. Then we get

$$(L+\varepsilon)m(n) \geq B'(m(n)) - A'(m(n)),$$
$$(L-\varepsilon)n \leq B'(n) - A'(n).$$

Hence $n(L-\varepsilon) \leq B'(n) - A'(n) \leq B'(m(n)) - A'(m(n)) \leq (L+\varepsilon)m(n)$ and

$$m(n) \geq n\frac{L-\varepsilon}{L+\varepsilon},$$
$$n - m(n) \leq \frac{2\varepsilon n}{L+\varepsilon} \leq \frac{2\varepsilon}{L}n.$$

We have $A'(n) - D'(n) \leq A'(n) - D'(m(n)) = A'(n) - A'(m(n)) \leq n - m(n)$,

$$0 \leq \frac{A'(n) - D'(n)}{n} \leq \frac{2\varepsilon}{L}$$

and

$$\lim_{n\to\infty} \frac{A'(n) - D'(n)}{n} = 0.$$

□

Of course the claim of this lemma holds also if $d(A) = d(B)$. We proved in fact also the following result:

Lemma 4. *If $A \cap B = \emptyset$, $\lim\limits_{n\to\infty} \frac{B(n)-A(n)}{n}$ exists and $\lim\limits_{n\to\infty} \frac{B(n)-A(n)}{n} > 0$, then there is a subset $D \subseteq B$ with $\lim\limits_{n\to\infty} \frac{D(n)-A(n)}{n} = 0$. In particular, $B \setminus D \in \mathcal{D}$.*

Corollary 4. *If $A, B \in \mathcal{D}$, $d(A) < d(B)$, then there exists $D \in \mathcal{D}$ such that $A \subseteq D \subseteq A \cup B$, $D \in \mathcal{D}$ and $d(D) = d(B)$.*

Proof. We have $d(\mathbb{N} \setminus A) > d(\mathbb{N} \setminus B)$. By Lemma 3 there exists a set $E \in \mathcal{D}$ such that $\mathbb{N} \setminus (A \cup B) \subseteq E \subseteq \mathbb{N} \setminus A$ and $d(E) = d(\mathbb{N} \setminus B)$. If we put $D = \mathbb{N} \setminus E$, then $A \subseteq D \subseteq A \cup B$ and $d(D) = d(B)$. □

Lemma 5. *Let $A \subseteq \mathbb{N}$. Then there exists a subset $B \subseteq A$ such that $B \in \mathcal{D}$ and $d(B) = \underline{\underline{d}}(A)$.*

Similarly, there exists a superset $C \supseteq B$ such that $C \in \mathcal{D}$ and $d(C) = \overline{\overline{d}}(A)$.

Proof. By the definition of $\underline{\underline{d}}(A)$ we have $\underline{\underline{d}}(A) = \sup\{d(B);\, B \subseteq A,\, B \in \mathcal{D}\}$. By results of Grekos and Volkmann [16] the density set $S(A)$ of all density points $(\underline{d}B, \overline{d}B)$, $B \subseteq A$, is closed, hence it contains its accumulation point $\big(\underline{\underline{d}}(A), \underline{\underline{d}}(A)\big)$. This point corresponds to the desired subset B of A.

The proof of the second part is analogous. □

Lemma 6. *If $A \cap B = \emptyset$, $A \in \mathcal{D}$, $\underline{\underline{d}}(B) = 0$, then $\underline{\underline{d}}(A \cup B) = d(A)$.*

Proof. Assume that $\underline{\underline{d}}(A\cup B) > d(A)$. Then there is $C \subseteq A\cup B$ with $d(C) > d(A)$. By Corollary 4 we may assume that $C \supseteq A$. Then $C \setminus A \in \mathcal{D}$, $d(C \setminus A) = d(C) - d(A) > 0$ and therefore $\underline{\underline{d}}(B) > 0$, a contradiction. $\square$

Lemma 7. *If $A \in \mathcal{D}$, $A \cap B = \emptyset$, then $\underline{\underline{d}}(A \cup B) = d(A) + \underline{\underline{d}}(B)$.*

Proof. By Lemma 5 there exists $B_1 \subseteq B$ such that $d(B_1) = \underline{\underline{d}}B$. Clearly, $\underline{\underline{d}}(B \setminus B_1) = 0$. Then using Lemma 6 we get $\underline{\underline{d}}(A \cup B) = \underline{\underline{d}}(A \cup B_1 \cup (B \setminus B_1)) = d(A \cup B_1) = d(A) + d(B_1) = d(A) + \underline{\underline{d}}(B)$. $\square$

Lemma 8. *If $B, C \in \mathcal{D}$ and $A \cup B \supseteq C$, then $d(C) - d(B) \le \underline{\underline{d}}(A)$.*

Proof. $d(C) \le \underline{\underline{d}}(A \cup B) = \underline{\underline{d}}((A \setminus B) \cup B) = \underline{\underline{d}}(A \setminus B) + d(B) \le \underline{\underline{d}}(A) + d(B)$. $\square$

We can see that the expression from Lemma 8 appears also in the definition of d_* (it is equal to $\sum_{i=1}^{p} d(A_i) - \sum_{j=1}^{q} d(B_j)$ for a special case $p = q = 1$). To prove Theorem 3 it suffices to show that every such a difference of two sums can be transformed to this simple case.

Let $A = \{a_1 < a_2 < \ldots\}$ be infinite and $m \in \mathbb{N}$. Define a set $B = \{b_1 < b_2 < \ldots\}$, where b_i is an arbitrary number from the set $\{ma_i + 1, ma_i + 2, \ldots, ma_i + m\}$. We will call the set of this kind an *m-copy of A*. Then it is easy to see that $\underline{\underline{d}}(A) = m \cdot \underline{\underline{d}}(B)$. We have also $\lim\limits_{n\to\infty} \frac{A(n)}{B(n)} = m$, thus $d(A) = m \cdot d(B)$ whenever $A \in \mathcal{D}$. Let us note that by [26, Theorem 1] $\mu(A) = m \cdot \mu(B)$ holds for any density measure as well.

Proof. [Proof of Theorem 3] Let

$$k\chi_A + \sum_{j=1}^{q} \chi_{B_j} \ge \sum_{i=1}^{p} \chi_{A_i}. \tag{3.4}$$

Put $l = \sum_{j=1}^{q} \chi_{B_j}$ and $r = \sum_{i=1}^{p} \chi_{A_i}$. Let $m \ge \max\limits_{n\in\mathbb{N}} (k\chi_A(n) + l(n))$. Taking

$$\begin{aligned} C &= \bigcup_{n\in\mathbb{N}} \{mn+1, mn+2, \ldots, mn+l(n)\} \\ D &= \bigcup_{n\in A} \{mn+l(n)+1, mn+l(n)+2, \ldots, mn+l(n)+k\}, \\ E &= \bigcup_{n\in\mathbb{N}} \{mn+1, mn+2, \ldots, mn+r(n)\} \end{aligned}$$

we get from (3.4) $C \cup D \supseteq E$.

The sets C, D, E can be viewed as a disjoint union of m-copies of the sets $B_1, B_2, \ldots, B_q$, a disjoint union of k m-copies of the set A and a disjoint union of m-copies of the sets $A_1, A_2, \ldots, A_p$, respectively. Hence we have $d(C) = \frac{1}{m} \sum\limits_{i=1}^{q} d(B_i)$,

$\underline{\underline{d}}(D)=\frac{k}{m}\cdot\underline{\underline{d}}(A)$ and $d(E)=\frac{1}{m}\sum\limits_{i=1}^{p}d(A_i)$. Thus by Lemma 8:

$$d(E)-d(C)\le\underline{\underline{d}}(D),$$
$$\frac{\sum_{i=1}^{n}d(A_i)-\sum_{j=1}^{m}d(B_j)}{m}\le\frac{k}{m}\cdot\underline{\underline{d}}(A),$$
$$\frac{\sum_{i=1}^{n}d(A_i)-\sum_{j=1}^{m}d(B_j)}{k}\le\underline{\underline{d}}(A).$$

Hence, $d_*(A)\le\underline{\underline{d}}(A)$. From (3.2) we have the reverse inequality, so we get $d_*(A)=\underline{\underline{d}}(A)$. The dual equality $d^*(A)=\overline{\overline{d}}(A)$ follows from $d_*(\mathbb{N}\setminus A)=\underline{\underline{d}}(\mathbb{N}\setminus A)$. □

The simplification obtained in Theorem 3 applied to the results of [4, Proposition 3.2.8] yields the following:

Proposition 1. *If $A,B\subset\mathbb{N}$ and $A\cap B=\emptyset$, then*

$$\underline{\underline{d}}(A)+\underline{\underline{d}}(B)\le\underline{\underline{d}}(A\cup B)\le\underline{\underline{d}}(A)+\overline{\overline{d}}(B)\le\overline{\overline{d}}(A\cup B)\le\overline{\overline{d}}(A)+\overline{\overline{d}}(B).$$

If $A\cap B=\emptyset$ and $A\cup B\in\mathcal{D}$, then

$$d(A\cup B)=\underline{\underline{d}}(A)+\overline{\overline{d}}(B).$$

If $A\in\mathcal{D}$, $A\cap B=\emptyset$, then

$$\overline{\overline{d}}(A\cup B)=d(A)+\overline{\overline{d}}(B).$$

It is easy to find examples showing that the above inequalities can be strict.

As an application of the lemmas used in the proof of Theorem 3 we prove some other interesting properties of $\underline{\underline{d}}$ and $\overline{\overline{d}}$ and of density measures.

Proposition 2. *If $A\subseteq B$ and $B\in\mathcal{D}$, then there exists $C\in\mathcal{D}$ such that $d(C)=\overline{\overline{d}}(A)$ and $A\subseteq C\subseteq B$.*

Similarly, if $A\subseteq B$ and $A\in\mathcal{D}$, then there exists $C\in\mathcal{D}$ such that $d(C)=\underline{\underline{d}}(B)$ and $A\subseteq C\subseteq B$.

Proof. By Lemma 5 there exists $D\in\mathcal{D}$ such that $d(D)=\overline{\overline{d}}(A)$ and $A\subseteq D$. Clearly, $d(D)=\overline{\overline{d}}(A)\le\overline{\overline{d}}(B)=d(B)$. By Lemma 3 there exists C such that $A\subseteq D\cap B\subseteq C\subseteq B$ and $d(C)=d(D)=\overline{\overline{d}}(A)$.

The second part is dual to the first one. □

Lemma 9. *If $A,B\subseteq\mathbb{N}$, and $\lim\limits_{n\to\infty}\frac{B(n)-A(n)}{n}$ exists and $\lim\limits_{n\to\infty}\frac{B(n)-A(n)}{n}>0$, then there is a set D such that $A\cap B\subseteq D\subseteq B$ with $\lim\limits_{n\to\infty}\frac{D(n)-A(n)}{n}=0$. In particular, $B\setminus D\in\mathcal{D}$.*

Proof. Use Lemma 4 for $B':=B\setminus A\cap B$ and $A':=A\setminus A\cap B$. □

Proposition 3. *Let $A, B \subseteq \mathbb{N}$. There exists the limit $\lim\limits_{n\to\infty} \frac{B(n)-A(n)}{n} = L$ if and only if $\mu(B)-\mu(A) = L$ for every density measure μ.*

Proof. If $\lim\limits_{n\to\infty} \frac{B(n)-A(n)}{n} = L$ then, by Lemma 9, there exists a subset $D \subset B$ with $\lim\limits_{n\to\infty} \frac{D(n)-A(n)}{n} = 0$. This implies $\mu(D) = \mu(A)$ for every density measure μ by [27, Proposition 3.3]. From this we get

$$\mu(B) = \mu(D) + \mu(B \setminus D) = \mu(A) + d(B \setminus D) = \mu(A) + L.$$

On the other hand, if $\mu(B) - \mu(A) = L$ for each density measure μ, then also

$$\mathcal{F}\text{-}\lim \frac{B(n)-A(n)}{n} = L$$

for every free ultrafilter $\mathcal{F}$. This implies that the only cluster point of the sequence $\left(\frac{B(n)-A(n)}{n}\right)$ is L and

$$\lim_{n\to\infty} \frac{B(n)-A(n)}{n} = L.$$

□

Remark 1. The functions $\underline{\underline{d}}$ and $\overline{\overline{d}}$ were studied also by Pólya [24] in a more general setting. He has studied sequences of non-negative real numbers such that the difference of successive elements is bounded from 0. We will use his result only for sequences of natural numbers. Among other things he proved in [24, Satz VIII] that

$$\underline{\underline{d}}(A) = \lim_{\theta\to 1^-} \liminf_{n\to\infty} \frac{A(n)-A(\theta n)}{n-\theta n}, \qquad \overline{\overline{d}}(A) = \lim_{\theta\to 1^-} \limsup_{n\to\infty} \frac{A(n)-A(\theta n)}{n-\theta n}.$$

Pólya called these values minimal and maximal density (Minimaldichte and Maximaldichte). This expression of $\underline{\underline{d}}$ and $\overline{\overline{d}}$ can serve as a basis for a different proof of Corollary 3.

Finitely additive measures on $\mathbb{N}$ can be understood as positive normed functionals on ℓ_∞. Clearly, if we identify a subset of $\mathbb{N}$ with its characteristic sequence, such a functional yields a measure on $\mathbb{N}$. The functional corresponding to a measure is in fact the integral with respect to this measure (obtained by imitating the definition of Riemann integral, see e.g. [30, Section 3]). A more detailed exposition into representation of finitely additive measures as the elements of the dual space ℓ_∞^* can be found in [6]. From the positivity and $\mu(\mathbb{N}) = 1$ we see that the norm of each measure on $\mathbb{N}$ in ℓ_∞^* is equal to 1.

Suppose we are given a set $A \subseteq \mathbb{N}$. Now, for a given $\theta < 1$, choose a sequence (n_i) such that

$$\lim_{i\to\infty} \frac{A(n_i)-A(\theta n_i)}{n_i-\theta n_i} = \liminf_{n\to\infty} \frac{A(n)-A(\theta n)}{n-\theta n}.$$

Let $\mathcal{F}$ be any free ultrafilter containing the set $\{n_i; i \in \mathbb{N}\}$. Then

$$\mu_\theta(B) = \mathcal{F}\text{-lim}\,\frac{B(n) - B(\theta n)}{n - \theta n} = \mathcal{F}\text{-lim}\left(\frac{B(n)}{n}\frac{n}{n-\theta n} + \frac{B(\theta n)}{\theta n}\left(1 - \frac{n}{n - \theta n}\right)\right)$$

defines a density measure on $\mathbb{N}$ such that

$$\mu_\theta(A) = \liminf_{n\to\infty} \frac{A(n) - A(\theta n)}{n - \theta n}.$$

We consider all these measures as the elements of the unit ball of ℓ_∞^*. By Banach-Alaoglu theorem this ball is compact in weak* topology. Let us choose a sequence μ_{θ_k} such that $\lim\limits_{k\to\infty} \theta_k = 1$ and $0 < \theta_k < 1$. Then the sequence (μ_{θ_k}) has a subsequence which is convergent in the weak* topology. We denote this subsequence by (μ_n) and the limit by μ. The convergence in weak* topology implies that

$$\mu(A) = \lim_{n\to\infty} \mu_n(A) = \underline{\underline{d}}(A).$$

The proof that there exists a density measure with $\mu(A) = \overline{\overline{d}}(A)$ is similar. Together with the convexity of the set of density measures and the obvious estimates $\underline{\underline{d}}(A) \le \mu(A) \le \overline{\overline{d}}(A)$ this yields Corollary 3.

4 Density measures with a given value for some set

Corollary 3 gives the complete answer to the question of Fey about the values of density measure. But this answer is only existential. The natural question arisen here is: *Which values are attained by density measures expressible in some "simple" form, e.g.* (2.1)*?*

Lemma 10. *Consider $\alpha \ge -1$ and $A \subseteq \mathbb{N}$. Then the set of all cluster points of the sequence $\left(\frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}\right)$ is the whole interval $\left[\underline{d_\alpha}(A), \overline{d_\alpha}(A)\right]$.*

Proof. The proof of this lemma is based on [1, Theorem 1] which claims that if a sequence (x_n) in a compact metric space (X, d) satisfies

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

then the set of all cluster points of (x_n) is connected. The connectedness of the set of cluster points of $\left(\frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}\right)$ is equivalent to the assertion of the lemma.

As $0 \le \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)} \le 1$ for all $n \in \mathbb{N}$, it suffices to show that $\lim\limits_{n\to\infty} \left|\frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)} - \frac{A_\alpha(n+1)}{\mathbb{N}_\alpha(n+1)}\right| = 0$. Assume that $\alpha > -1$. Then we have

$$\begin{aligned}\left|\frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)} - \frac{A_\alpha(n+1)}{\mathbb{N}_\alpha(n+1)}\right| &\le \left|\frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)} - \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n+1)}\right| + \frac{(n+1)^\alpha}{\mathbb{N}_\alpha(n+1)} \\ &= \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}\left|1 - \frac{\mathbb{N}_\alpha(n)}{\mathbb{N}_\alpha(n+1)}\right| + \frac{(n+1)^\alpha}{(n+1)^{\alpha+1}}\,\frac{(n+1)^{\alpha+1}}{\mathbb{N}_\alpha(n+1)}\end{aligned}$$

Now, using Lemma 2, it can be easily seen that $\lim\limits_{n\to\infty}\left|\frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}-\frac{A_\alpha(n+1)}{\mathbb{N}_\alpha(n+1)}\right|=0$.

Analogous treatment can be used also for $\alpha=-1$. The only difference is replacing the term $(n+1)^{\alpha+1}$ by $\ln(n+1)$ in the last part of the above estimation. $\square$

Corollary 5. *Let $A\subseteq\mathbb{N}$ and $\alpha\geq -1$. For every $x\in\left[\underline{d_\alpha}(A),\overline{d_\alpha}(A)\right]$ there is a free ultrafilter $\mathcal{F}$ such that $\mu_\alpha^{\mathcal{F}}(A)=x$.*

Proof. According to Lemma 10, x is a cluster point of the sequence $\left(\frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}\right)$. Hence there is an (infinite) set $K=\{n_1<n_2<\dots\}$ such that $x=\lim\limits_{k\to\infty}\frac{A_\alpha(n_k)}{\mathbb{N}_\alpha(n_k)}$. Taking any free ultrafilter $\mathcal{F}$ containing the set K one can easily show that $\mu_\alpha^{\mathcal{F}}(A)=x$. This completes the proof. $\square$

The following result follows from Rajagopal [25].

Theorem 4. *Let $-1\leq\alpha\leq\beta$. Then for all $A\subseteq\mathbb{N}$ we have*

$$\underline{d_\beta}(A)\leq\underline{d_\alpha}(A)\leq\overline{d_\alpha}(A)\leq\overline{d_\beta}(A).$$

This led us to introduce the following notation:

$$\underline{d_\infty}(A)=\lim_{\alpha\to\infty}\underline{d_\alpha}(A)=\inf_{\alpha\geq-1}\underline{d_\alpha}(A);$$
$$\overline{d_\infty}(A)=\lim_{\alpha\to\infty}\overline{d_\alpha}(A)=\sup_{\alpha\geq-1}\overline{d_\alpha}(A).$$

Theorem 5. *If $A\subseteq\mathbb{N}$ and $x\in\left(\underline{d_\infty}(A),\overline{d_\infty}(A)\right)$, then there is a density measure μ of the form (2.1) such that $\mu(A)=x$.*

Proof. By definition of $\underline{d_\infty}(A)$ and $\overline{d_\infty}(A)$ there is an $\alpha\geq-1$ such that $x\in\left[\underline{d_\alpha}(A),\overline{d_\alpha}(A)\right]$. The rest follows from Corollary 5. $\square$

Using Theorem 2 we get

Corollary 6. *For all $A\subseteq\mathbb{N}$ we have*

$$\underline{\underline{d}}(A)\leq\underline{d_\infty}(A)\leq\overline{d_\infty}(A)\leq\overline{\overline{d}}(A).$$

Using Corollary 3 and Corollary 6 we are able to show that the expression (2.1) does not describe all density measures. There are also density measures of different type:

Again, taking the set $A=\bigcup\limits_{k=0}^{\infty}\left(2^{2k},2^{2k+1}\right]\cap\mathbb{N}$ we have $\underline{\underline{d}}(A)=\underline{d_\infty}(A)=0$. On the other hand, $\mu_\alpha^{\mathcal{F}}(A)\geq\frac{1}{2^{\alpha+1}+1}>0$. Hence $\mu(A)=\int_\Omega\mu_\alpha^{\mathcal{F}}(A)\,\mathrm{d}\psi(\mathcal{F},\alpha)>0$ for any probability Borel measure ψ, too. But by Corollary 3 there is also a density measure μ' with $\mu'(A)=\underline{\underline{d}}(A)=0$.

We recall the definition of gap density and some results from [16]. The value of the gap density $\lambda(A)$ describes how large gaps can be between elements of A. It is given by

$$\lambda(A) = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$$

for $A = \{a_1 < a_2 < a_3 < \ldots\}$. The sets having infinite gap density are called thin sets in [5].

It is shown in [16] that the density set $S(A) = \{(\underline{d}(B), \overline{d}(B)); B \subseteq A\}$ is located above the line $y = \lambda(A)x$. It follows, that if $\lambda(A) > 1$, then $\overline{d}(B) \geq \lambda(A)\underline{d}(B) > \underline{d}(B)$ for any subset B of A with $\underline{d}(B) > 0$ (see also [15, Proposition 2.1]). Hence no subset of A has density strictly greater than 0 and $\underline{\underline{d}}(A) = 0$.

On the other hand, if $\lambda(A) > 1$, then there are arbitrary large n's with $a_{n+1} > (1+\varepsilon)a_n + 1$. Thus if $b_n = \lceil (1+\varepsilon)a_n \rceil$ we get $A_\alpha(b_n) = A_\alpha(a_n)$ and $b_n \geq (1+\varepsilon)a_n$. Hence $(\alpha+1)\frac{A_\alpha(b_n)}{b_n^{\alpha+1}} = (\alpha+1)\frac{A_\alpha(a_n)}{b_n^{\alpha+1}} \leq \frac{1}{(1+\varepsilon)^{\alpha+1}} \frac{(\alpha+1)\mathbb{N}_\alpha(a_n)}{{a_n}^{\alpha+1}}$ and by Lemma 2 we get $\underline{d_\alpha}(A) \leq \frac{1}{(1+\varepsilon)^{\alpha+1}}$. Consequently $\underline{d_\infty}(A) = 0$.

Proposition 4. *If $\lambda(A) > 1$, then $\underline{\underline{d}}(A) = \underline{d_\infty}(A) = 0$.*

The question of the equality of $\underline{\underline{d}}(A)$ and $\underline{d_\infty}(A)$ for a set A with $\lambda(A) = 1$ remains open.

Problem 1. *Is it true that $\underline{\underline{d}}(A) = \underline{d_\infty}(A)$ for every $A \subseteq \mathbb{N}$?*

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