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Statistical convergence and $\mathcal{I}$-convergence

Abstract

In this paper we introduce the concept of $\mathcal{I}$-convergence of sequences of real numbers based on the notion of the ideal of subsets of N. The $\mathcal{I}$-convergence gives a unifying look on several types of convergence related to the statistical convergence. In a sense it is equivalent to the concept of μ-statistical convergence introduced by J. Connor (μ being a two valued measure defined on a subfield of 2^N).

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Introduction

This paper comes out from the concept of statistical convergence which is introduced in [9] and [26] and further developed in other papers (cf e.g. [5], [6], [8], [11], [12], [13], [17]). Several new types of convergence of sequences appeared, many of them are related to the statistical convergence. The concept of $\mathcal{I}$-convergence gives a unifying approach to these types of convergence.

Definition and Notations

Recall the concept of asymptotic density of set $A \subseteq N$ (cf. [20], p. 71, 95-96).

If $A \subseteq N = \{1, 2, \ldots, n, \ldots\}$, then χ_A denotes characteristic function of the set A, i.e. $\chi_A(k) = 1$ if $k \in A$ and $\chi_A(k) = 0$ if $k \in N \setminus A$. Put $d_n(A) = \frac{1}{n}\sum_{k=1}^{n}\chi_A(k)$, $\delta_n(A) = \frac{1}{S_n}\sum_{k=1}^{n}\frac{\chi_A(k)}{k}$ $(n = 1, 2, \ldots)$, where $S_n = \sum_{k=1}^{n}\frac{1}{k}$ $(n = 1, 2, \ldots)$.

Then the numbers $\underline{d}(A) = \liminf_{n\to\infty} d_n(A)$, $\overline{d}(A) = \limsup_{n\to\infty} d_n(A)$ are called the lower and upper asymptotic density of A, respectively.

Similarly, the numbers $\underline{\delta}(A) = \liminf_{n\to\infty}\delta_n(A)$, $\overline{\delta}(A) = \limsup_{n\to\infty}\delta_n(A)$ are called the lower and upper logarithmic density of A, respectively.

If there exist $\lim_{n\to\infty} d_n(A) = d(A)$ and $\lim_{n\to\infty}\delta_n(A) = \delta(A)$ then $d(A)$ and $\delta(A)$ are called the asymptotic and logarithmic density of A, respectively.

It is well-known fact, that for each $A \subseteq N$

$$\underline{d}(A) \leq \underline{\delta}(A) \leq \overline{\delta}(A) \leq \overline{d}(A) \tag{1}$$

(cf. [20], p. 95).

Hence if exists $d(A)$, then $\delta(A)$ exists as well and $d(A) = \delta(A)$.

Note that number $\underline{d}(A)$, $\overline{d}(A)$, $d(A)$, $\underline{\delta}(A)$, $\overline{\delta}(A)$, $\delta(A)$ belong to the interval $[0, 1]$.

Owing to the well-known formula

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(\tfrac{1}{n}),\ n \to \infty, \tag{2}$$

(γ is the Euler constant), we can write $\ln n$ instead of S_n ($n > 1$) in the definition of $\delta_n(A)$ (cf [20], p. 45).

Recall the concept of statistical convergence (cf. [9], [26]):

Definition A. *A sequence $x = (x_n)_1^\infty$ of real numbers is said to be statistically convergent to $\xi \in R$ provided that for each $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where $A(\epsilon) = \{n \in N \ : \ |x_n - \xi| \geq \epsilon\}$.*

In what follows we will use the concept of an ideal and filter of sets.

Definition B. *Let $X \neq \emptyset$. A class $\mathcal{S} \subseteq 2^X$ of subsets of X is said to be an ideal in X provided that $\mathcal{S}$ is additive and hereditary , i.e if $\mathcal{S}$ satisfies these conditions:*

(i) $\emptyset \in \mathcal{S}$,

(ii) $A, B \in \mathcal{S} \Rightarrow A \cup B \in \mathcal{S}$,

(iii) $A \in \mathcal{S},\ B \subseteq A \Rightarrow B \in \mathcal{S}$

(cf. [14], p. 34).

An ideal is called *non-trivial* if $X \notin \mathcal{S}$.

Definition C. *Let $X \neq \emptyset$. A non-empty class $\mathcal{F} \subseteq 2^X$ of subsets of X is said to be a filter in X provided that:*

(j) $\emptyset \notin \mathcal{F}$,

(jj) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$,

(jjj) $A \in \mathcal{F},\ B \supseteq A \Rightarrow B \in \mathcal{F}$

(cf. [18], p. 44).

The following proposition expresses a relation between the notions of ideal and filter:

Proposition A. *Let $\mathcal{S}$ be a non-trivial ideal in X, $X \neq \emptyset$. Then the class*

$$\mathcal{F}(\mathcal{S}) = \{M \subseteq X : \exists A \in \mathcal{S} : M = X \setminus A\}$$

is a filter on X (we will call $\mathcal{F}(\mathcal{S})$ the filter associated with $\mathcal{S}$).

The proof of Proposition A is easy and so it can be left to the reader.

Definition D. *A non-trivial ideal $\mathcal{S}$ in X is called admissible if $\{x\} \in \mathcal{S}$ for each $x \in X$.*

We will also use the concept of porosity of subsets of a metric space (cf. [27], pp. 183-212, [28]).

Let (Y,ρ) be a metric space, $M \subseteq Y$. Let $B(y,\delta)$ $(y \in Y, \delta >))$ denote the ball with centre y and radius δ, i.e. $B(y,\delta) = \{x \in Y : \rho(x,y) < \delta\}$. For $y \in Y$, $\delta > 0$ we put

$$\gamma(y,\delta,M) = \sup\{t > 0 : \exists z \in B(y,\delta) : [B(z,t) \subseteq B(y,\delta)] \wedge [B(z,t) \cap M = \emptyset]\}.$$

If such a $t > 0$ does not exist, then we put $\gamma(y,\delta,M) = 0$.

The numbers

$$\underline{p}(y,M) = \liminf_{\delta \to 0^+} \frac{\gamma(y,\delta,M)}{\delta}, \ \overline{p}(y,M) = \limsup_{\delta \to 0^+} \frac{\gamma(y,\delta,M)}{\delta}$$

are called the lower and upper porosity of set M at y. If for all $y \in Y$ we have $\overline{p}(y,M) > 0$ then M is said to be porous in Y. Obviously every set porous in Y is nowhere dense in Y.

If $\overline{p}(y,M) \geq c > 0$ then M is said to be c-porous at y and it is said to be c-porous in Y if it is c-porous at each $y \in Y$.

If $\underline{p}(y,M) > 0$ then M is said to be very porous at y. If M is very porous at y for each $y \in Y$, then M is said to be very porous in Y. The concept of very c-porous set at y and very c-porous set in y can be defined analogously. If $\underline{p}(y,M) = \overline{p}(y,M)(= p(y,M))$ then the number $p(y,M)$ is called the porosity of M at y. If $p(y,M) = 1$ then M is said to be strongly porous at y.

The paper is divided into four sections. In the first one the concept of $\mathcal{I}$-convergence is introduced and its fundamental properties are studied. It is shown here that this concept gives a unifying approach to many various types of convergence related to statistical convergence.

In the second section fundamental arithmetical properties of this convergence are established.

In the third section a convergence (so called $\mathcal{I}^*$-convergence) is introduced. This is a convergence parallel to $\mathcal{I}$-convergence . Necessary and sufficient conditions are given for equivalence of these two types of convergence.

In the fourth section the convergence fields of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence are investigated.

1 $\mathcal{I}$-convergence of sequences of real numbers - examples

The concept of statistical convergence and the study of similar types of convergence (cf. [3], [4], [17], [26]) lead us to introducing the notion of $\mathcal{I}$-convergence of sequences . This notion gives a unifying look at many types of convergence related to statistical convergence.

Definition 1.1 *Let $\mathcal{I}$ be a non-trivial ideal in N. A sequence $x = (x_n)_1^\infty$ of real numbers is said to be $\mathcal{I}$-convergent to $\xi \in R$ if for every $\epsilon > 0$ the set $A(\epsilon) = \{n : |x_n - \xi| \geq \epsilon\}$ belongs to $\mathcal{I}$.*

If $x = (x_n)_1^\infty$ is $\mathcal{I}$-convergent to ξ we write $\mathcal{I} - \lim x_n = \xi$ (or $\mathcal{I} - \lim x = \xi$) and the number ξ is called the $\mathcal{I}$-limit of $x = (x_n)_1^\infty$.

A question arises whether the concept of $\mathcal{I}$-convergence satisfies some usual axioms of convergence (cf. [16]). The most known axioms of convergence are the following axioms (formulated for $\mathcal{I}$-convergence):

(S) Every stationary sequence $x = (\xi, \xi, \ldots, \xi, \ldots)$ $\mathcal{I}$-converges to ξ.

(H) The uniqueness of limit: If $\mathcal{I} - \lim x = \xi$ and $\mathcal{I} - \lim x = \eta$, then $\xi = \eta$.

(F) If $\mathcal{I} - \lim x = \xi$, then for each subsequence y of x we have $\mathcal{I} - \lim y = \xi$.

(U) If each subsequence y of a sequence x has a subsequence z $\mathcal{I}$-convergent to ξ, then x is $\mathcal{I}$-convergent to ξ.

Theorem 1.1 *Let $\mathcal{I}$ be an admissible ideal in N. Then*
(i) $\mathcal{I}$-convergence satisfies the axioms (S), (H) and (U).
(ii) if $\mathcal{I}$ contains an infinite set, then $\mathcal{I}$-convergence does not satisfy the axiom (F).

Remark. If an admissible ideal $\mathcal{I}$ contains no infinite set, then $\mathcal{I}$ coincides with the class of all finite subsets of N and the $\mathcal{I}$-convergence is equal to the usual convergence in R, therefore it satisfies the axiom (F) (see ideal $\mathcal{I}_f$ in (III) in what follows).

Proof of Theorem 1.1. It is obvious that $\mathcal{I}$-convergence ($\mathcal{I}$ being an admissible ideal) satisfies the axiom (S). We prove that it satisfies (H) as well.Suppose that $\mathcal{I} - \lim x_n = \xi$, $\mathcal{I} - \lim x_n = \eta$, $\xi \neq \eta$. Choose

$$\epsilon \in (0, \tfrac{|\xi - \eta|}{2}). \tag{3}$$

Then by assumption and Proposition A the sets $N \setminus A(\epsilon) = \{n : |x_n - \xi| < \epsilon\}$, $N \setminus B(\epsilon) = \{n : |x_n - \eta| < \epsilon\}$ belong to the filter $\mathcal{F}(\mathcal{I})$. But then the set $(N \setminus A(\epsilon)) \cap (N \setminus B(\epsilon))$ belongs to $\mathcal{F}(\mathcal{I})$, too. Hence there is an $m \in N$ such that $|x_m - \xi| < \epsilon$, $|x_m - \eta| < \epsilon$. From this $|\xi - \eta| < 2\epsilon$ which is a contradiction to (3).

We prove that $\mathcal{I}$-convergence satisfies the axiom (U). We prove the following statement equivalent to (U):

If $\mathcal{I} - \lim x_n = \xi$ does not hold, then there exist a subsequence y of x such that no subsequence z of y is $\mathcal{I}$-convergent to ξ.

By Definition 1.1 there exist an ϵ_0 such that

$$A(\epsilon_0) = \{n : |x_n - \xi| \geq \epsilon_o\} \notin \mathcal{I}. \tag{4}$$

Then $A(\epsilon_0)$ is an infinite set since $\mathcal{I}$ is an admissible ideal. Let

$$A(\epsilon_0) = \{n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots\} \subseteq N.$$

Put $y_k = x_{n_k}$, $(k = 1, 2, \ldots)$. Then $y = (y_k)_1^\infty$ is a subsequence of x and by (4):

$$|y_k - \xi| \geq \epsilon_0, \; (k = 1, 2, \ldots). \tag{4*}$$

From (4*) we see that no subsequence $z = (z_m)_1^\infty$ of y can be $\mathcal{I}$-convergent , since in the opposite case the set N would belong to $\mathcal{I}$.

(ii)Suppose that an infinite set $A = \{n_1 < n_2 < \ldots < n_k < \ldots\} \subseteq N$ belongs to $\mathcal{I}$. Put

$$B = N \setminus A = \{m_1 < m_2 < \ldots < m_k < \ldots\}.$$

The set B is again infinite because in the opposite case N would belong to $\mathcal{I}$.

Define the sequence $x = (x_n)_1^\infty$ as follows

$$x_{n_k} = 0,\ x_{m_k} = 1\ (k = 1, 2, \ldots).$$

Obviously $\mathcal{I} - \lim x_n = 1$.. Simultaneously the subsequence $y = (x_{n_k})_{k=1}^\infty$ of x is stationary and so $\mathcal{I} - \lim y = 0$ (see axiom (S)). Hence $\mathcal{I}$-convergence does not satisfy the axiom (F). □

In what follows we introduce several examples of $\mathcal{I}$-convergence .

(I) Put $\mathcal{I}_0 = \{\emptyset\}$. This is the minimal non-empty non-trivial ideal in N. Obviously a sequence is $\mathcal{I}_0$-convergent if and only if it is constant.

(II) Let $\emptyset \neq M \subseteq N$, $M \neq N$. Put $\mathcal{I}_M = 2^M$. Then $\mathcal{I}_M$ is a non-trivial ideal in N. A sequence $x = (x_n)_1^\infty$ is $\mathcal{I}_M$-convergent if and only if it is constant on $N \setminus M$, i.e. if there is a $\xi \in R$ such that $x_n = \xi$ for each $n \in N \setminus M$. (Obviously (I) is a special case of (II) for $M = \emptyset$.)

(III) Denote by $\mathcal{I}_f$ the class of all finite subsets of N. Then $\mathcal{I}_f$ is an admissible ideal in N and $\mathcal{I}_f$-convergence coincides with the usual convergence in R.

(IV) Put $\mathcal{I}_d = \{A \subseteq N : d(A) = 0\}$. Then $\mathcal{I}_d$ is an admissible ideal in N and $\mathcal{I}_d$-convergence coincides with the statistical convergence.

(V) Put $\mathcal{I}_\delta = \{A \subseteq N : \delta(A) = 0\}$. Then $\mathcal{I}_\delta$ is an admissible ideal in N and we will call the $\mathcal{I}_\delta$-convergence the logarithmic statistical convergence. If $\mathcal{I}_\delta - \lim x_n = \xi$ then $\mathcal{I}_d - lim x_n = \xi$ (see (1)). The converse is not true.

(VI) The examples (IV), (V) can be generalized. Choose $c_n > 0$, $(n = 1, 2, \ldots)$ such that $\sum_{n=1}^\infty c_n = +\infty$. Put

$$h_m(A) = \frac{\sum_{i \leq m, i \in A} c_i}{\sum_{i=1}^m c_i},\ (m = 1, 2, \ldots).$$

Denote by $h(A)$ the limit $\lim_{m\to\infty} h_m(A)$ (if it exists) (cd. [1]). Then $\mathcal{I}_h = \{A \subseteq N : h(A) = 0\}$ is an admissible ideal in N and $\mathcal{I}_d$-convergence and $\mathcal{I}_\delta$-convergence are special cases of I_h-convergence.

(VII) Denote by $u(A)$ the uniform density of the set A (cf [2]). then $I_u = \{A \subseteq N : u(A) = 0\}$ is an admissible ideal and $\mathcal{I}_u$-convergence will be called the uniform statistical convergence.

(VIII) A wide class of $\mathcal{I}$-convergences can be obtained in the following manner: Let $T = (t_{nk})$ be a non-negative regular matrix (cf. [21], p. 8). Then for each $A \subseteq N$ the series

$$d_T^{(n)}(A) = \sum_{k=1}^\infty t_{nk} \chi_A(k)\ (n = 1, 2, \ldots)$$

converge. If there exists

$$d_T(A) = \lim_{n\to\infty} d_T^{(n)}(A)$$

then $d_T(A)$ is called the T-density of A (cf. [17]). By the regularity of T we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} = 1$$

and from this we can see that $d_T(A) \in [0,1]$. Put $\mathcal{I}_{d_T} = \{A \subseteq N : d_T(A) = 0\}$. Then $\mathcal{I}_{d_T}$ is an admissible ideal in N and $\mathcal{I}_{d_T}$-convergence contains as special case the $\mathcal{I}_h$-convergence from (VI) (choosing $t_{nk} = \frac{c_k}{\sum_{i=1} nc_i}$ for $k \le n$, $t_{nk} = 0$ for $k > n$, $n = 1, 2, \ldots$).

The $\mathcal{I}_{d_T}$-convergence includes also the ϕ-convergence of Schoenberg (cf. [26]) (choosing $t_{nk} = \frac{\phi(k)}{n}$ for $k \le n$, k divides n, $t_{nk} = 0$ for $k \le n$, k does not divide n and $t_{nk} = 0$ for $k > n$, $n = 1, 2, \ldots$), ϕ being the Euler function.

(IX) Let υ be a finite additive measure defined on a class $\mathcal{U}$ of subsets of N (cf. [7], [10], [15], [23], [24], [25]) which contain all finite subsets of N and $\upsilon(\{n\}) = 0$ for each $n \in N$, $\upsilon(A) \le \upsilon(B)$ if $A, B \in \mathcal{U}$, $A \subseteq B$. Then $\mathcal{I}_\upsilon = \{A \subseteq N : \upsilon(A) = 0\}$ is an admissible ideal in N. The $\mathcal{I}_d$- and $\mathcal{I}_\delta$-convergences are included in $\mathcal{I}_\upsilon$-convergence. Further for υ we can take the measure density of R. C. Buck (cf. [3]).

(X) Let $\mu_m : 2^N \to [0,1]$, $m = 1, 2, \ldots$ be finitely additive measures defined on 2^N. If there exists $\mu(A) = \lim_{m\to\infty} \mu_m(A)$, then $\mu(A)$ is called the measure of A. Obviously μ is a finitely additive measure defined on a class $\mathcal{E} \subseteq 2^N$, so $\mathcal{I}_\mu = \{A \subseteq N : \mu(A) = 0\}$ is an admissible ideal in N. For μ_m we can take d_m, δ_m (see Definitions and Notations).

(XI) Let $N = \bigcup_{j=1}^{\infty} D_j$ be a decomposition of N (i.e. $D_k \cap D_l = \emptyset$ for $k \ne l$), assume that D_j $(j = 1, 2, \ldots)$ are infinite sets (e.g we can choose $D_j = \{2^{j-1}(2s-1) : s = 1, 2 \ldots\}$). Denote by $\mathcal{J}$ the class of all $A \subseteq N$ such that A intersects only a finite number of D_j. Then it is easy to see that $\mathcal{J}$ is an admissible ideal in N.

(XII) In [11] the concept of density ρ of sets $A \subseteq N$ is axiomatically introduced. Using this concept we can define the ideal $\mathcal{I}_\rho = \{A \subseteq N : \rho(A) = 0\}$ and obtain $\mathcal{I}_\rho$-convergence as a generalization of statistical convergence.

(XIII) We introduce yet the following admissible ideal $\mathcal{I}_c$ in N connected with the convergence of subseries of the harmonic series : $\mathcal{I}_c = \{A \subseteq N : \sum_{a\in A} a^{-1} < +\infty\}$ (for $A = \emptyset$ we put $\sum_{a\in A} a^{-1} = 0$). Then from $\mathcal{I}_c$-convergence the $\mathcal{I}_d$-convergence follows since $\mathcal{I}_c \subset \mathcal{I}_d$ (cf [22]).

In the end of this section we introduce some remarks on the relation between our $\mathcal{I}$-convergence and μ-statistical convergence of J. Connor (cf. [7]). The approach of J. Connor to generalization of statistical convergence is based on using a finite additive measure μ defined on a field Γ of subsets of N with $\mu(\{k\}) = 0$ for each $k \in N$ and such that $A, B \in \Gamma$, $A \subseteq B$ implies $\mu(A) \le \mu(B)$. Put

$$\mathcal{I} = \{A \in \Gamma : \mu(A) = 0\}.$$

Then it is easy to verify that $\mathcal{I}$ is an admissible ideal in N and $\mathcal{F}(\mathcal{I}) = \{B \subseteq N : \mu(B) = 1\}$.

Conversely, if $\mathcal{I}$ is an admissible ideal in N, then we put

$$\Gamma = \mathcal{I} \cup \mathcal{F}(\mathcal{I}).$$

Then Γ is a field (algebra) of subsets of N. Define $\mu : \Gamma \to \{0, 1\}$ as follows:

$$\mu(M) = 0 \text{ if } M \in \mathcal{I},$$

$$\mu(M) = 1 \text{ if } M \in \mathcal{F}(\mathcal{I}).$$

This definition is correct since it is easy to see that $\mathcal{I} \cap \mathcal{F}(\mathcal{I}) = \emptyset$. Further $\mu(\{k\}) = 0$ since $\mathcal{I}$ is admissible, the monotonity and additivity of μ can be easily checked.

Hence these two approaches to generalization of statistical convergence seem to be equivalent in such a sense that each of them can be replaced (by the method mentioned above) by other approach.

2 Fundamental arithmetical properties of $\mathcal{I}$-convergence

We show that $\mathcal{I}$-convergence has many arithmetical properties similar to properties of the usual convergence.

Theorem 2.1 *Let $\mathcal{I}$ be a non-trivial ideal in N.*

(i) If $\mathcal{I} - \lim x_n = \xi$, $\mathcal{I} - \lim y_n = \eta$, then $\mathcal{I} - \lim(x_n + y_n) = \xi + \eta$.

(ii) If $\mathcal{I} - \lim x_n = \xi$, $\mathcal{I} - \lim y_n = \eta$, then $\mathcal{I} - \lim(x_n y_n) = \xi\eta$.

(iii) If $\mathcal{I}$ is an admissible ideal in N, then $\lim_{n\to\infty} x_n = \xi$ implies $\mathcal{I} - \lim x_n = \xi$.

Proof . (i) Let $\epsilon > 0$. Then the inclusion

(5) $$\{n : |x_n + y_n - (\xi + \eta)| \geq \epsilon\} \subseteq \{n : |x_n - \xi| \geq \tfrac{\epsilon}{2}\} \cup \{n : |y_n - \eta| \geq \tfrac{\epsilon}{2}\}$$

can be easily verified. The sets on the right-hand side of (5) belong to $\mathcal{I}$. By Definition B the set on the left-hand side of (5) belongs to $\mathcal{I}$, too.

(ii) Let $\epsilon > 0$. The following inclusion can be checked.

(6) $$\{n : |x_n y_n - \xi\eta| < \epsilon\} = \{n : |x_n(y_n - \eta) + \eta(x_n - \xi)| < \epsilon\} \supseteq \{n : \{|x_n| < |\xi| + 1\} \cap \{n : |y_n - \eta| < \tfrac{\epsilon}{2(|\xi|+1}\} \cap \{n : |x_n - \xi| < \tfrac{\epsilon}{2|\eta|}\}.$$

Observe that if $|x_n - \xi| < 1$, then $|x_n| = |x_n - \xi + \xi| \leq |x_n - \xi| + |\xi| < 1 + |\xi|$. Consequently, if $\frac{\epsilon}{2|\eta|} \leq 1$

$$\{n : |x_n| < |\xi| + 1\} \supseteq \{n : |x_n - \xi| < 1\} \supseteq \{n : |x_n - \xi| < \frac{\epsilon}{2|\eta|}\}.$$

So from (6) we get

(6*) $\{n : |x_n y_n - \xi\eta| < \epsilon\} \supseteq \{n : |x_n - \xi| < \frac{\epsilon}{2|\eta|}\} \cap \{n : |y_n - \eta| < \frac{\epsilon}{2(|\xi|+1)}\}$.

By the assumption each of the sets on the right-hand side of (6*) belongs to $\mathcal{F}(\mathcal{I})$, from this it easily follows that the set on the left-hand side of (6*) belongs to $\mathcal{F}(\mathcal{I})$. But then its complement $\{n : |x_n y_n - \xi\eta| \geq \epsilon\}$ belongs to $\mathcal{I}$.

(*iii*) This part of theorem follows from the fact that $\mathcal{I}_f$ is contained as a subset in every admissible ideal. □

3 $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence

In connection with Definition 1.1 we introduce yet another type of convergence which corresponds to convergence in μ-density of J. Connor (cf. [7])

Definition 3.1 *Let $\mathcal{I}$ be an admissible ideal in N. A sequence $x = (x_n)_1^\infty$ of real numbers is said to be $\mathcal{I}^*$-convergent to $\xi \in R$ (shortly $\mathcal{I}^* - \lim x_n = \xi$ or $\mathcal{I}^* - \lim x = \xi$) if there is a set $H \in \mathcal{I}$, such that for $M = N \setminus H = \{m_1 < m_2 < \ldots\}$ we have*

$$\lim_{k\to\infty} x_{m_k} = \xi. \tag{7}$$

Remark. We write also $\lim_{n\to\infty, n\in M} x_n = \xi$ instead of (7).

We now have two types of "ideal convergence". For every admissible ideal $\mathcal{I}$ the following relation between them holds:

Theorem 3.1 *Suppose that $\mathcal{I}$ is an admissible ideal in N. If $\mathcal{I}^* - \lim x_n = \xi$ then $\mathcal{I} - \lim x_n = \xi$.*

Proof . By assumption there is a set $H \in \mathcal{I}$ such that (7) holds, where $M = N \setminus H = \{m_1 < m_2 < \ldots\}$. Let $\epsilon > 0$. By (7) there is a $k_o \in N$, such that $|x_{m_k} - \xi| < \epsilon$ for $k > k_0$. Put $A(\epsilon) = \{n : |x_n - \xi| \geq \epsilon\}$. Then

$$A(\epsilon) \subseteq H \cup \{m_1, m_2, \ldots, m_{k_0}\}. \tag{8}$$

Since $\mathcal{I}$ is admissible and $H \in \mathcal{I}$, the union on the right-hand side of (8) belongs to $\mathcal{I}$ and so $A(\epsilon) \in \mathcal{I}$. □

Remark. For some ideals the converse of Theorem 3.1 holds (e.g for $\mathcal{I}_d$ in (IV), cf [23]). In [10] it is proved that (in our terminology) the converse of Theorem 3.1. does not hold for $\mathcal{I}_u$-convergence (see (VII)). We now give a new example of such an ideal.

Example 3.1. Put $\mathcal{I} = \mathcal{J}$ (see (XI)). Define $x = (x_n)_1^\infty$ as follows: For $n \in D_j$ we put $x_n = \frac{1}{j}$ $(j = 1, 2, \ldots)$. Then obviously $\mathcal{I} - \lim x_n = 0$. But we show that $\mathcal{I}^* - \lim x_n = 0$ does not hold.

If namely $H \in \mathcal{I}$ then there is (by definition of $\mathcal{I}$) a $p \in N$ such that

$$H \subseteq D_1 \cup D_2 \cup \ldots \cup D_p.$$

But then $D_{p+1} \subseteq N \setminus H$ and so by notation used in proof of Theorem 3.1. we have $x_{m_k} = \frac{1}{p+1}$ for infinitely many of k's. Therefore $\lim_{k\to\infty} x_{m_k} = 0$ cannot be true.

In what follows we give a necessary and sufficient condition for equivalency of $\mathcal{I}$ and $\mathcal{I}^*$-convergences. This condition is similar to the condition (APO) in [7], [10].

Definition 3.2 *An admissible ideal $\mathcal{I}$ in N is said to satisfy the condition (AP) if for every countable system $\{A_1, A_2, \ldots\}$ of mutually disjoint sets belonging to $\mathcal{I}$ there exist sets $B_j \subseteq N$ $(j = 1, 2, \ldots)$ such that the symmetric differences $A_j \div B_j$ $(j = 1, 2, \ldots)$ are finite and $B = \bigcup_{j=1}^{\infty} B_j$ belongs to $\mathcal{I}$.*

Remark. Observe that each B_j from previous definition belongs to $\mathcal{I}$.

Theorem 3.2 *From $\mathcal{I} - \lim x_n = \xi$ the statement $\mathcal{I}^* - \lim x_n = \xi$ follows if and only if $\mathcal{I}$ satisfies the condition (AP).*

Corollary 3.1 *From Theorems 3.1. and 3.2 we obtain : The equivalency $\mathcal{I} - \lim x_n = \xi \Leftrightarrow \mathcal{I}^* - \lim x_n = \xi$ holds if and only if $\mathcal{I}$ satisfies the condition (AP).*

Proof of Theorem 3.2. 1) Suppose that $\mathcal{I}$ satisfies the condition (AP).

Let $\mathcal{I} - \lim x_n = \xi$. Then for every $\epsilon > 0$ the set $A(\epsilon) = \{n : |x_n - \xi| \geq \epsilon\}$ belongs to $\mathcal{I}$.

Consequently each of the following sets A_j $(j = 1, 2, \ldots)$ belongs to $\mathcal{I}$

$$A_1 = \{n : |x_n - \xi| \geq 1\} = A(1),$$

$$A_k = \{n : \frac{1}{k} \leq |x_n - \xi| < \frac{1}{k-1}\} = A(\frac{1}{k}) \setminus A(\frac{1}{k-1}).$$

Obviously $A_i \cap A_j = \emptyset$ for $i \neq j$.

Since $\mathcal{I}$ satisfies (AP) there are sets $B_j \subseteq N$ such that $A_j \div B_j$ is a finite set $(j = 1, 2, \ldots)$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

It suffices to prove that

$$\lim_{n\to\infty,\ n\in M} x_n = \xi, \tag{9}$$

where $M = N \setminus B$.

Let $\eta > 0$. Choose a $k \in N$ such that $\frac{1}{k+1} < \eta$. Then

$$\{n : |x_n - \xi| \geq \eta\} \subseteq \bigcup_{j=1}^{k+1} A_j.$$

The set on the right hand-side belongs to $\mathcal{I}$ by the additivity of $\mathcal{I}$. Since $A_j \div B_j$ is finite $(j = 1, 2, \ldots, k+1)$, there is an $n_0 \in N$, such that

$$\bigcup_{j=1}^{k+1} B_j \cap (n_0, +\infty) = \bigcup_{j=1}^{k+1} A_j \cap (n_0, +\infty).$$

If now $n \notin B$, $n > n_0$ then $n \notin \bigcup_{j=1}^{k+1} B_j$ and so $n \notin \bigcup_{j=1}^{k+1} A_j$. But then $|x_n - \xi| < \frac{1}{k+1} < \eta$. Hence (9) hold.

2) Suppose that from $\mathcal{I} - \lim x_n = \xi$ the statement $\mathcal{I}^* - \lim x_n = \xi$ follows. We prove that $\mathcal{I}$ fulfils the condition (AP).

Let $\{A_1, A_2, \ldots\}$ be a class of disjoint sets of $\mathcal{I}$. Define $x = (x_n)_1^\infty$ as follows $x_n = \frac{1}{j}$ for $n \in A_j$ $(j = 1, 2, \ldots)$, $x_n = 0$ for $n \in N \setminus \bigcup_j A_j$.

First of all we show that $\mathcal{I} - \lim x_n = 0$.

Let $\epsilon > 0$. Choose an m such that $\frac{1}{m} < \epsilon$. Then

$$A(\epsilon) = \{n : |x_n - \xi| \geq \epsilon\} \subseteq A_1 \cup A_2 \cup \ldots \cup A_m.$$

From this we see that $A(\epsilon) \in \mathcal{I}$, hence $\mathcal{I} - \lim x_n = 0$. Consequently by the assumption we have

$$\mathcal{I}^* - \lim x_n = 0.$$

But then there is a set $B \in \mathcal{I}$ such that

$$\lim_{n\to\infty, n\in N\setminus B} x_n = 0. \tag{10}$$

Put $B_j = A_j \cap B$, $(j = 1, 2, \ldots)$. It suffices to show that $A_j \div B_j$ $(j = 1, 2, \ldots)$ is finite. Indeed if this is true, then

$$\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} (B \cap A_j) = B \cap \bigcup_{j=1}^{\infty} A_j \subseteq B.$$

Since $B \in \mathcal{I}$, we see that $\bigcup_j B_j \in \mathcal{I}$.

Put $N \setminus B = \{m_1 < m_2 < \ldots\}$. Then by (10) $\lim_{k\to\infty} x_{m_k} = 0$. From this we see that the set A_j has only a finite number of elements common with the set $N \setminus B$.

$$A_j \div B_j \subseteq A_j \cap (N \setminus B),$$

we have $A_j \div B_j$ is finite. □

4 Convergence fields of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence

The $\mathcal{I}$-convergence and also $\mathcal{I}^*$-convergence can be considered as a summability methods that (in the case of admissibility of $\mathcal{I}$) are regular. Denote by $F(\mathcal{I})$ and $F(\mathcal{I}^*)$ the convergence field of $\mathcal{I}$- and $\mathcal{I}^*$-convergence , respectively. Hence in detail

$$F(\mathcal{I}) = \{x = (x_n) \in \ell_\infty : \text{ there exists } \mathcal{I} - \lim x_n \in R\},$$

$$F(\mathcal{I}^*) = \{x = (x_n) \in \ell_\infty : \text{ there exists } \mathcal{I}^* - \lim x_n \in R\}.$$

These convergence fields will be studied in what follows as subsets of the linear normed space ℓ_∞ of all bounded real sequence with the sup -norm

$$\|x\| = \sup_{n=1,2,\dots} |x_n|, \ x = (x_n)_1^\infty \in \ell_\infty.$$

As we have already seen, $F(\mathcal{I})$ is a linear subspace of ℓ_∞ (see Theorem 2.1.). The similar fact for $F(\mathcal{I}^*)$ is obvious.

We shall see that the properties of $F(\mathcal{I})$ and $F(\mathcal{I}^*)$ depend on the ideal $\mathcal{I}$. therefore we shall study at first some properties of ideals, mainly maximality of ideals in N.

The class $\mathcal{Z}$ of all admissible ideals in N can be partially ordered by inclusion. If $\mathcal{Z}_0 \subseteq \mathcal{Z}$ is a non-empty (linearly) ordered subclass of $\mathcal{Z}$, then it can be easily checked that $\bigcup \mathcal{Z}_0$ is again an admissible ideal in N, which is an upper bound for $\mathcal{Z}_0$. Thus we can use the Zorn lemma which gives the existence of a maximal ideal in $\mathcal{Z}$.

We give in what follows a characterization of maximal admissible ideals in N.

Lema 4.1 *An ideal $\mathcal{I}_0$ in N is a maximal admissible ideal in N if and only if for every $A \subseteq N$ following statement holds:*

$$(V) \qquad (A \in \mathcal{I}_0) \vee (N \setminus A \in \mathcal{I}_0).$$

Proof . 1. Suppose that $\mathcal{I}_0$ satisfies (V). We show that $\mathcal{I}_0$ is maximal admissible. Suppose in contrary that there is an admissible ideal $\mathcal{I}_1$ in N such that $\mathcal{I}_1 \supset \mathcal{I}_0$. Then there is a set $A \subseteq N$, such that $A \in \mathcal{I}_1 \setminus \mathcal{I}_0$. Hence $A \notin \mathcal{I}_0$ and consequently by (V) we have $N \setminus A \in \mathcal{I}_0$. But then $A \in \mathcal{I}_1$, $N \setminus A \in \mathcal{I}_1$ which gives $N \in \mathcal{I}_1$- a contradiction.

2. Suppose that $\mathcal{I}_0$ is a maximal admissible ideal. We prove (V). We proceed indirectly. Then there is a set $A \subseteq N$ such that we have:

$$(11) \qquad (A \notin \mathcal{I}_0) \wedge (N \setminus A \notin \mathcal{I}_0).$$

Construct the class $\mathcal{K} = \{X \subseteq N : X \cap A \in \mathcal{I}_0\}$. We show that

a) $\mathcal{K} \supseteq \mathcal{I}_0$,

b) $\mathcal{K}$ is an admissible ideal in N.

a) Let $X \in \mathcal{I}_0$. Then $X \cap A \subseteq X$. Therefore $X \cap A \in \mathcal{I}_0$ and so $X \in \mathcal{K}$.

b) Evidently $N \notin \mathcal{K}$ and $\mathcal{K}$ contains each singleton $\{n\}$, $n \in N$.

Further if X_1, $X_2 \in \mathcal{K}$, then $X_1 \cap A$, $X_2 \cap A \in \mathcal{I}_0$, therefore $(X_1 \cup X_2) \cap A = (X_1 \cap A) \cup (X_2 \cap A)$ belongs to $\mathcal{I}_0$ and $X_1 \cup X_2$ belongs to $\mathcal{K}$.

Let $X \in \mathcal{K}$ and $X_1 \subseteq X$. Then $X_1 \cap A \subseteq X \cap A \in \mathcal{I}_0$, hence $X_1 \cap A \in \mathcal{I}_0$, $X_1 \in \mathcal{K}$.

So we have proved that $\mathcal{K}$ is an admissible ideal in N and $\mathcal{K} \supseteq \mathcal{I}_0$. By maximality of $\mathcal{I}_0$ we have $\mathcal{I}_0 = \mathcal{K}$. Observe that $N \setminus A \in \mathcal{K}$ as $(N \setminus A) \cap A = \emptyset \in \mathcal{I}_0$. But this contradicts (11). $\square$

We return to linear subspaces $F(\mathcal{I})$, $F(\mathcal{I}^*)$. The "magnitude" of $F(\mathcal{I})$ depends very simply on $\mathcal{I}$. This fact is shown in the following statement:

Theorem 4.1 *Let $\mathcal{I}$ be an admissible ideal in N. Then $F(\mathcal{I}) = \ell_\infty$ if and only if $\mathcal{I}$ is a maximal admissible ideal in N.*

Proof . 1. Suppose that $\mathcal{I}$ is a maximal admissible ideal in N. Let $x = (x_n)_1^\infty \in \ell_\infty$. We show that there exists $\mathcal{I} - \lim x_n \in R$.

Since $x \in \ell_\infty$ there are numbers $a, b \in R$ such that $a \leq x_n \leq b$, $(n = 1, 2, \ldots)$. Put $A_1 = \{n : a \leq x_n \leq \frac{a+b}{2}\}$, $B_1 = \{n : \frac{a+b}{2} \leq x_n \leq b\}$. Then $A_1 \cup B_1 = N$. Since $\mathcal{I}$ is an admissible ideal both sets A_1, B_1 cannot belong to $\mathcal{I}$. Thus at least one of them does not belong to $\mathcal{I}$. Denote it by D_1 and interval corresponding to it denote by I_1. So we have (infinite) set D_1 and interval I_1 such that $D_1 = \{n : x_n \in I_1\} \notin \mathcal{I}$.

So we can (by induction) construct a sequence of closed intervals $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$, $I_n = [a_n, b_n]$, $(n = 1, 2, \ldots$, $\lim_{n\to\infty}(a_n - b_n) = 0$ and sets $D_k = \{n : x_n \in I_k\} \notin \mathcal{I}$ $(k = 1, 2, \ldots)$.

Let $\xi \in \bigcap_{k=1}^\infty I_k$ and $\epsilon > 0$. Construct the set $M(\epsilon) = \{n : |x_n - \xi| < \epsilon\}$. For sufficiently large m we have $I_m = [a_m, b_m] \subseteq (\xi - \epsilon, \xi + \epsilon)$. Since $D_m \notin \mathcal{I}$ we see that $M(\epsilon) \notin \mathcal{I}$.

Since $M(\epsilon) \notin \mathcal{I}$, the maximality of $\mathcal{I}$ implies (see Lemma 4.1.) that $A(\epsilon) = \{n : |x_n - \xi| \geq \epsilon\} = N \setminus M(\epsilon) \in \mathcal{I}$. Hence $\mathcal{I} - \lim x_n = \xi$.

2. Suppose that $\mathcal{I}$ is not maximal. Then by Lemma 4.1. there is a set $M = \{m_1 < m_2 < \ldots\} \subseteq N$ such that $M \notin \mathcal{I}$, $N \setminus M \notin \mathcal{I}$. Define the sequence $x = (x_n)_1^\infty$ as follows: $x_n = \chi_M(n)$ $(n = 1, 2, \ldots)$. Then $x_n \in \ell_\infty$. We show that $\mathcal{I} - \lim x_n$ does not exist. This follows from the fact that for every $\xi \in R$ and sufficiently small ϵ the set $\{n : |x_n - \xi| \geq \epsilon\}$ is equal to M or $N \setminus M$ or to whole N and neither of these belongs to $\mathcal{I}$. □

Remark. The previous theorem cannot be extended for unbounded sequences. This is shown in the following:

Proposition 4.1 *Let $\mathcal{I}$ be an admissible ideal. Then there exists an unbounded sequence of real numbers for which $\mathcal{I} - \lim x$ does not exist.*

Proof . It is easy to see that sequence $x_n = n$, $n = 1, 2, \ldots$ is a wanted example. □

In what follows we will deal with topological properties of convergence fields $F(\mathcal{I})$, $F(\mathcal{I}^*)$ and the relation between them.

Theorem 4.2 *Suppose that $\mathcal{I}$ is an admissible ideal in N. Then $F(\mathcal{I})$ is a closed linear subspace of ℓ_∞.*

Proof . Let $x^{(m)} = (x_j^{(m)})_{j=1}^\infty \in F(\mathcal{I})$ $(m = 1, 2, \ldots)$, $\lim_{m\to\infty} x^{(m)} = x$, $x = (x_j)_1^\infty$ in ℓ_∞, i.e. $\lim_{m\to\infty} \|x^{(m)} - x\| = 0$. We prove that $x \in F(\mathcal{I})$.

By the assumption there exist $\mathcal{I} - \lim x^{(m)} = \xi_m \in R$, $(m = 1, 2, \ldots)$.

The proof will be realized in two steps:

1. We prove that $(\xi_m)_1^\infty$ is a Cauchy sequence (so that there exists $\lim_{m\to\infty} \xi_m = \xi \in R$).

2. We prove that $\mathcal{I}-\lim x=\xi$.

1) Let $\eta>0$. From $\lim_{m\to\infty}x^{(m)}=x$ we deduce that $(x^{(m)})_1^\infty$is a Cauchy sequence in ℓ_∞. Therefore there is an $m_0\in N$ such that for each u, $v>m_0$ we have

$$\|x^{(u)}-x^{(v)}\|<\tfrac{\eta}{3}. \tag{12}$$

Fix u, $v>m_0$. Note that sets $U(\frac{\eta}{3})=\{j:|x_j^{(u)}-\xi_u|<\frac{\eta}{3}\}$, $V(\frac{\eta}{3})=\{j:|x_j^{(v)}-\xi_v|<\frac{\eta}{3}\}$ belong to $\mathcal{F}(\mathcal{I})$, thus their intersection is non-void. For element $s\in U(\frac{\eta}{3})\cap V(\frac{\eta}{3})$ we have

$$|x_s^{(u)}-\xi_u|<\tfrac{\eta}{3},\ |x_s^{(v)}-\xi_v|<\tfrac{\eta}{3}. \tag{13}$$

By a simple estimate we get from (12), (13):

$$|\xi_u-\xi_v|\le|\xi_u-x_s^{(u)}|+|x_s^{(u)}-x_s^{(v)}|+|x_s^{(v)}-\xi_v|<\frac{\eta}{3}+\frac{\eta}{3}+\frac{\eta}{3}=\eta.$$

Hence $(\xi_m)_1^\infty$ is a Cauchy sequence and so there exists $\xi=\lim_{m\to\infty}\xi_m\in R$.

2) Let $\epsilon>0$. Choose v_0 such that for $v>v_0$ we have simultaneously

$$|\xi_v-\xi|<\tfrac{\epsilon}{3}|,\ \|x^{(v)}-x\|\le\tfrac{\epsilon}{3}. \tag{14}$$

For each $n\in N$ we have

$$|x_n-\xi|\le|x_n-x_n^{(v)}|+|x_n^{(v)}-\xi_v|+|\xi_v-\xi|. \tag{15}$$

Let $A(\epsilon)=\{n:|x_n-\xi|\ge\epsilon\}$, $CA(\epsilon)=\{n:|x_n-\xi|<\epsilon\}$, $A_v(\frac{\epsilon}{3})=\{n:|x_n^{(v)}-\xi_v|\ge\frac{\epsilon}{3}\}$, $CA_v(\frac{\epsilon}{3})=\{n:|x_n^{(v)}-\xi_v|<\frac{\epsilon}{3}\}$. Then by (14), (15) we get $|x_n-\xi|<\epsilon$ for every $n\in A_v(\frac{\epsilon}{3})$. So we have

$$CA_v(\tfrac{\epsilon}{3})\subseteq CA(\epsilon). \tag{16}$$

Note that $A_v(\frac{\epsilon}{3})\in\mathcal{I}$. If we take complements of sets in (16) we find out that $A(\epsilon)\in\mathcal{I}$. The proof of 2) is finished. □

We can summarize our results concerning the convergence fields $F(\mathcal{I})$, $F(\mathcal{I}^*)$. From Theorem 3.1. we know that $F(\mathcal{I}^*)\subseteq F(\mathcal{I})$ and Theorem 3.2. says that the equality $F(\mathcal{I})=F(\mathcal{I}^*)$ holds if and only if $\mathcal{I}$ satisfies the condition (AP). Further $F(\mathcal{I})=\ell_\infty$ if and only if $\mathcal{I}$ is a maximal ideal (see Theorem 4.1.). Thus if $\mathcal{I}$ is not maximal and does not satisfy the condition (AP) then

$$F(\mathcal{I}^*)\subset F(\mathcal{I})\subset\ell_\infty.$$

Now we show that for every admissible ideal $\mathcal{I}$ the set $F(\mathcal{I}^*)$ is dense in $F(\mathcal{I})$.

Theorem 4.3 *For every admissible ideal $\mathcal{I}$ in N we have*

$$\overline{F(\mathcal{I}^*)}=F(\mathcal{I}).$$

($\overline{M}$ is the closure of M in ℓ_∞.) **Proof .** By Theorem 3.1. we have $F(\mathcal{I}^*) \subseteq F(\mathcal{I})$. Since $F(\mathcal{I})$ is closed in ℓ_∞, we get $\overline{F(\mathcal{I}^*)} \subseteq F(\mathcal{I})$. Hence it suffices to prove that

$$F(\mathcal{I}) \subseteq \overline{F(\mathcal{I}^*)}.$$

Put $B(z,\delta) = \{x \in \ell_\infty : \|x - z\| < \delta\}$ for $z \in \ell_\infty$, $\delta > 0$ (ball in ℓ_∞). It suffices to prove that for each $y \in F(\mathcal{I})$ and $0 < \delta < 1$ we have

$$B(y,\delta) \cap F(\mathcal{I}^*) \neq \emptyset. \tag{17}$$

Put $L = \mathcal{I} - \lim y$. Choose an arbitrary $\eta \in (0,\delta)$. Then

$$A(\eta) = \{n : |y_n - L| \geq \eta\} \in \mathcal{I}.$$

Define $x = (x_n)_1^\infty$ as follows: $x_n = y_n$ if $n \in A(\eta)$ and $x_n = L$ otherwise.
Then obviously $x \in \ell_\infty$, $\mathcal{I}^* - \lim x = L$ and $x \in B(y,\eta)$. So (17) holds. □

It is well known fact that if W is a closed linear subspace of a linear normed space X and $X \neq W$, then W is a nowhere dense set in X (cd. [19], Lemma 1). This fact evokes the question about the porosity of W. We will give the answer to this question in general and show some applications to convergence fields $F(\mathcal{I})$ and $F(\mathcal{I}^*)$.

Lema 4.2 *Suppose that X is a linear normed space and W is its closed linear subspace, $W \neq X$. Let*

$$s(W) = \sup\{\delta > 0 : \exists B(y,\delta) \subseteq B(Q,1) \setminus W\}$$

(Q being the zero element of X). Then $s(W) = \frac{1}{2}$.

Proof . We proceed indirectly. Suppose that $s(W) > \frac{1}{2}$. Then there is a $\delta > \frac{1}{2}$ such that for suitable y we have

$$B(y,\delta) \subseteq B(Q,1) \setminus W. \tag{18}$$

There are two possible cases:

$$(\alpha)\ \|y\| > \frac{1}{2}, \quad (\beta)\|y\| \leq \frac{1}{2}.$$

(α) In this case for every c, $\frac{1}{2} < c < \delta$ we have

$$y + \frac{c}{\|y\|} y \in B(y,\delta).$$

Simultaneously we have

$$\|y + \frac{c}{\|y\|} y\| = (1 + \frac{c}{\|y\|})\|y\| = \|y\| + c > 1.$$

Hence $y + \frac{c}{\|y\|} y \notin B(Q,1)$, which contradicts (18).

(β) In this case $Q \in B(y,\delta) \cap W$ which contradicts (18) too.

Hence $s(W) \le \frac{1}{2}$.

Let $v \in X \setminus W$ and put $\alpha = \inf_{u \in W} \|v - u\|$. Obviously $\alpha > 0$. Without loss of generality we can assume $\alpha = \frac{1}{2}$ (if $\alpha \neq \frac{1}{2}$ we can take $\frac{1}{2\alpha} v$ instead of v). For $\epsilon \in (0, \frac{1}{2})$ there exists $u_0 \in W$ such that $\frac{1}{2} \le \|v - u_0\| < \frac{1}{2} + \epsilon$ by definition of α. Put $y = v - u_0$ and $\delta = \frac{1}{2} - \epsilon$. We show that (18) holds for them.

If $z \in B(y,\delta)$ then $\|z - y\| < \frac{1}{2} - \epsilon$ and $\|z\| \le \|z - y\| + \|y\| < (\frac{1}{2} - \epsilon) + (\frac{1}{2} + \epsilon) = 1$, i.e $B(y,\delta) \subseteq B(Q,1)$.

Suppose that $z \in B(y,\delta) \cap W$ then $\|z - y\| < \frac{1}{2} - \epsilon$. On the other hand $\|z - y\| = \|z - (v - u_0)\| = \|(z + u_0) - v\| \ge \frac{1}{2}$, since $z + v_0 \in W$. We get a contradiction, hence $B(y,\delta) \cap W = \emptyset$.

If $\epsilon \to 0^+$ then $\delta \to \frac{1}{2}^-$ and we get $s(W) = \frac{1}{2}$. □

Theorem 4.4 *Suppose that X is a linear normed space and W is its closed linear subspace, $W \neq X$. Then W is a very porous set in X, in more detail*

a) If $x \in X \setminus W$ then $p(x, W) = 1$,

b) If $x \in W$ then $p(x, W) = \frac{1}{2}$.

Proof . The part *a*) is an easy consequence of the closedness of W in X.

We prove *b*). Since $W \neq X$, there is a $u \in B(Q,1) \setminus W$ and $\delta > 0$ such that

$$B(u,\delta) \subseteq B(Q,1) \setminus W. \tag{19}$$

First we show that

$$\|u\| + \delta \le 1. \tag{20}$$

We proceed indirectly. Assume that $\|u\| + \delta > 1$. Since $\|u\| < 1$ for a suitable $c > 0$ we have $1 < \|u\| + c\|u\| < \|u\| + \delta$. From this $c\|u\| < \delta$ and so

$$u + cu \in B(u,\delta). \tag{21}$$

On the other hand

$$\|u + cu\| = (1 + c)\|u\| = \|u\| + c\|u\| > 1$$

and so $u + cu \notin B(Q,1)$, which contradicts (19), (21). Hence (20) holds.

Let $x \in W$, $\epsilon > 0$. We show that

$$B(x + \epsilon u, \epsilon\delta) \subseteq B(x,\epsilon) \setminus W, \tag{22}$$

if (19) holds.

For $z \in B(x + \epsilon u, \epsilon\delta)$ we put $w = z - x - \epsilon u$. Then

$$\|w\| = \|z - x - \epsilon u\| < \epsilon\delta. \tag{23}$$

Further $z - x = \epsilon u + w$, hence by (20), (23)

$$\|z - x\| = \|\epsilon u + w\| \leq \|\epsilon u\| + \|w\| < \epsilon\|u\| + \epsilon\delta \leq \epsilon.$$

From this we get $z \in B(x, \epsilon)$.

We show yet $z \notin W$. In the contrary case we have $z - x = \epsilon u + w \in W$, hence

$$u + \tfrac{1}{\epsilon} w \in W. \tag{24}$$

Since $\|\frac{1}{\epsilon} w\| < \delta$ (see (23)), $\frac{1}{\epsilon} w \in B(u, \delta)$. But then by (19) we get $u + \frac{1}{\epsilon} w \notin W$, which contradicts (24).

Hence we have proved the inclusion (22) under the assumption that $B(u, \delta) \subseteq B(Q, 1) \setminus W$. But then by definition of $\gamma(x, \epsilon, W)$ we have $\gamma(x, \epsilon, W) \geq \epsilon\delta$ for each $\delta > 0$ such that (19) holds. From this we get $\gamma(x, \epsilon, w) \geq \epsilon s(W)$,

$$\underline{p}(x, W) \geq s(W),$$

where $s(W) = \frac{1}{2}$ is introduced in Lemma 4.2.

Since for every ball $B(y, \delta)$, $\delta > \frac{1}{2}$, $B(y, \delta) \subseteq B(Q, 1)$ we have $Q \in B(y, \delta)$, the assertion of Theorem 4.4 follows from Lemma 4.2. □

We will apply Theorem 4.4 to the study of the structure of convergence fields $F(\mathcal{I})$, $F(\mathcal{I}^*)$, $\mathcal{I}$ being an admissible ideal in N. We take the linear normed space ℓ_∞ of all bounded real sequences with the sup-norm

$$\|x\| = \sup_{n=1,2,\ldots} |x_n|, \ x = (x_n)_1^\infty \in \ell_\infty.$$

By Theorem 4.1. the convergence field $F(\mathcal{I})$ coincides with ℓ_∞ if and only if $\mathcal{I}$ is a maximal ideal. Hence it is convenient to deal with $F(\mathcal{I})$ under the assumption that $\mathcal{I}$ is not maximal. In this case we have $F(\mathcal{I}) \subset \ell_\infty$ and by Theorem 4.2 the set $F(\mathcal{I})$ is a closed linear subspace of ℓ_∞.

The following theorem is an easy consequence of Theorem 4.4.

Theorem 4.5 *Suppose that $\mathcal{I}$ is an admissible ideal in N which is not maximal. Then the following holds:*

1. *If $x \in \ell_\infty \setminus F(\mathcal{I})$, then $p(x, F(\mathcal{I})) = 1$.*
2. *If $x \in F(\mathcal{I})$, then $p(x, F(\mathcal{I})) = \frac{1}{2}$.*

Since $F(\mathcal{I}^*) \subseteq F(\mathcal{I}) = \overline{(F(\mathcal{I}^*)}$ (see Theorem 3.1., Theorem 4.3.), we get

Corollary 4.1 *Under the condition of Theorem 4.5. we have:*

1. *If $x \in \ell_\infty \setminus F(\mathcal{I})$, then $p(x, F(\mathcal{I}^*)) = 1$.*
2. *If $x \in F(\mathcal{I})$, then $p(x, F(\mathcal{I}^*)) = \frac{1}{2}$.*

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