# Regular maps from Cayley graphs III: $t$-balanced Cayley maps*

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# Overview

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# Maps

- *map*: 2-cell embedding $G \to S$ of a graph $G$ into a surface $S$.

- combinatorial description: $\mathcal{M} = (K; R, L)$, where
  $K$: connected graph,
  $R$: permutation of $D(K)$ (dart set of $K$),
  $L$: involution of $D(K)$,
  $\langle R, L \rangle$ transitive on $D(K)$.

- Aut($\mathcal{M}$) acts semiregularly on $D(K)$.

- *regular map*: Aut($\mathcal{M}$) acts transitively (i.e, regularly) on $D(K)$.

# Cayley Maps

Let $G$ be a group, $\Omega$ be a generating set s.t. $1_g \notin \Omega = \Omega^{-1}$ and $p$ be a cyclic permutation of $\Omega$.

- *Cayley graph* $\mathrm{C}(G, \Omega)$:

  vertex set: $G$,

  dart set: $G \times \Omega$

  incidence: $g \xrightarrow{(g,x)} gx$

- *Cayley map* $\mathrm{CM}(G, \Omega, p)$: $(\mathrm{C}(G, \Omega); R, L)$, where

$$R(g, x) = (g, p(x)) \quad \text{and} \quad L(g, x) = (gx, x^{-1}).$$

# History

1972 Biggs
1992 Širáň, Škoviera - balanced Cayley maps
1994 Škoviera, Širáň - antibalanced Cayley maps
1998 Martino, Schultz - $t$-balanced Cayley maps
2002 Jajcay, Širáň - skew morphisms
???? Richter, Širáň, Jajcay, Tucker, Watkins - Cayley maps

Martino, Schultz, Škoviera : Cayley maps and $t$-automorphisms (preprint)
Conder, Jajcay, Tucker : Regular $t$-balanced Cayley maps (JCT B)

# $t$-balanced Cayley maps

Cayley map $\mathcal{M} = \text{CM}(G, \Omega, p)$ is

- *balanced* if $p(x^{-1}) = (p(x))^{-1}$,
- *antibalanced* if $p(x^{-1}) = (p^{-1}(x))^{-1}$, and
- $t$-*balanced* if $p(x^{-1}) = (p^t(x))^{-1}$.

Necessarily, $t^2 \equiv 1 \pmod{|\Omega|}$.

Note:

$$\text{balanced} \Leftrightarrow \text{1-balanced},$$
$$\text{antibalanced} \Leftrightarrow -1\text{-balanced}.$$

## Distribution of inverses

Let $\mathcal{M} = \mathrm{CM}(G, \Omega, p)$ be a Cayley map and let $x_1, \dots x_n$ be an ordering of $\Omega$ such that $p(x_i) = x_{i+1}$.

- *distribution of inverses* $\tau$: $x_i^{-1} = x_{\tau(i)}$.

- balanced maps:

| | | |
|---|---|---|
| $\tau(i) = u + i$ | involution-free | $(n = 2u)$ |
| $\tau(i) = i$ | all involutions | |

- antibalanced maps:

| | | |
|---|---|---|
| $\tau(i) = 1 - i$ | involution-free | $(n = 2u)$ |
| $\tau(i) = -i$ | 2 involutions | $(n = 2u)$ |
| $\tau(i) = -i$ | 1 involution | $(n = 2u + 1)$ |

**Theorem.** *Let $\mathcal{M} = CM(G, \Omega, p)$ be a $t$-balanced Cayley map. Then there exists an ordering $x_1, x_2, \ldots, x_n$ of the generating set $\Omega$ such that $p(x_i) = x_{i+1}$ and such that the distribution of inverses $\tau$ has one of the following forms:*

$$(i)\ \tau(i) = ti, \text{ or}$$

*(if $n$ is even, $2^k || n$, $t \equiv \pm 1 \pmod{2^k}$ and $d = \gcd(t-1, n)/2$)*

$$(ii)\ \tau(i) = d + ti,$$

Case $(i)$: $\gcd(t-1, n)$ involutions,
Case $(ii)$: involution-free.

- $t$-balanced maps:
  $\tau(i) = ti$ STANDARD
  $\tau(i) = d + ti$ EXCEPTIONAL

- balanced maps:
  $\tau(i) = u + i$ EXCEPTIONAL
  $\tau(i) = i$ STANDARD

- antibalanced maps:
  $\tau(i) = 1 - i$ EXCEPTIONAL
  $\tau(i) = -i$ STANDARD

## Groups with sign structure

Širáň, Škoviera 1992:

Let $G$ be a group with a generating set $\Omega$.

- *sign structure*: homomorphism $|\ | : G \to \{-1, 1\}$ with a kernel

$$G^+ = \{g \in G;\ g = y_1 \dots y_{2n},\ y_i \in \Omega,\ n \in \mathbb{N}_0\}.$$

- *trivial* sign structure: $G = G^+$.

**Theorem.** *Let $\mathcal{M} = CM(G, \Omega, p)$ be a regular $t$-balanced Cayley map. Then either $\Omega$ induces a nontrivial sign structure on $G$ or $\mathcal{M}$ is balanced.*

# $t$-automorphisms

Martino, Schultz 1998:
Let $G$ be a group with sign structure and $t \in \mathbb{Z} \setminus \{0\}$.

- *$t$-automorphism*: $\varphi \in \mathrm{Sym}(G)$, $|\varphi(g)| = |g|$ and

$$\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h),$$

($\pi : G \to \{1, t\}$, $\pi(g) = 1$, for $g \in G^+$, $\pi(g) = t$, for $g \notin G^+$).

Note: $t^2 \equiv 1 \pmod{\mathrm{ord}(\varphi)}$.

**Theorem.** *Let $\mathcal{M} = CM(G, \Omega, p)$ be a Cayley map such that $\Omega$ induces a nontrivial sign structure on $G$. Then $\mathcal{M}$ is regular and $t$-balanced if and only if there exists a $t$-automorphism $\varphi$ of $G$ whose restriction to $\Omega$ is equal to $p$.*

Let $G$ be a group with sign structure. Let $\varphi$ be a $t$-automorphism of $G$, $\xi = \varphi|_{G^+}$ and $w = \varphi(f)f^{-1}$ for some $f \notin G^+$. Then $\xi \in \mathrm{Aut}(G^+)$, $w \in G^+$ and

$$\varphi(hf^i) = \xi(h)w^i f^i.$$

**Theorem.** *Let $G$ be a group with sign structure. Let $\xi \in \mathrm{Aut}(G^+)$, $w \in G^+$ and $f \notin G^+$ be given, and put $\overline{w} = \xi^{-t}(\xi^{t-1}(w)\xi^{t-2}(w)\ldots\xi(w)w)$. Then the mapping $\varphi : hf^i \mapsto \xi(h)w^i f^i$ is a $t$-automorphism of $G$ iff:*

$$\xi\nu_f(\overline{w}^{-1}f^2) = wf^2$$

$$\xi\nu_f = \nu_w\nu_f\xi^t = \nu_{ws}\nu_f^{-1}\xi^t.$$

*($\nu_x$ is a conjugation by $x$.)*

# Exponents

Nedela, Škoviera 1997:

- *exponent* of a map $(K; R, L)$: integer $e$ s.t.$(K; R, L) \cong (K; R^e, L)$
(−1 is an exponent ⇔ map is reflexible).

- *exponent group* $\text{Ex}(\mathcal{M})$: exponents modulo valency.

**Theorem.** *Let* $\mathcal{M} = \text{CM}(G, \Omega, p)$ *be a regular* $t$*-balanced Cayley map of valency* $n$ *and* $e \in \mathbb{Z}_n^*$. *Then*

$$e \in \text{Ex}(\mathcal{M}) \Leftrightarrow \exists \sigma \in \text{Aut}(G, \Omega) : \ \sigma' p = p^e \sigma',$$

*where* $\sigma' = \sigma|_\Omega$.

# Examples

Complete bipartite graph $K_{n,n}$ with the standard regular embedding

$$\langle R, L; R^n = L^2 = (RL)^{2n} = 1,\ R(RL)^2 = (RL)^2 R \rangle$$

is a regular $t$-balanced Cayley map for each feasible $t$.

If $8|n$, then $K_{n,n}$ with embedding

$$\langle R, L; R^n = L^2 = (RL)^{2n} = 1,\ R(RL)^2 = (RL)^{2-n} R \rangle$$

is a regular $n/2 - 1$-balanced Cayley map.

Thank You