Regular t-balanced Cayley maps from cyclic and Abelian groups

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Maps

- *map*: 2-cell embedding $G \to S$ of a graph G into a surface S.

- combinatorial description: $\mathcal{M} = (K; R, L)$, where

  K: connected graph,

  R: permutation of $D(K)$ (dart set of K),

  L: involution of $D(K)$,

  $\langle R, L \rangle$ transitive on $D(K)$.

- Aut($\mathcal{M}$) acts semiregularly on $D(K)$.

- *regular map*: Aut($\mathcal{M}$) acts transitively (i.e, regularly) on $D(K)$.

Cayley Maps

Let G be a group, Ω be a generating set s.t. $e_G \notin \Omega = \Omega^{-1}$ and p be a cyclic permutation of Ω.

- *Cayley graph* $\mathsf{C}(G, \Omega)$:

  vertex set: G,
  dart set: $G \times \Omega$
  incidence: $g \xrightarrow{(g,x)} gx$

- *Cayley map* $\mathsf{CM}(G, \Omega, p)$: $(\mathsf{C}(G, \Omega); R, L)$, where

$$R(g, x) = (g, p(x)) \quad \text{and} \quad L(g, x) = (gx, x^{-1}).$$

t-balanced Cayley maps

Cayley map $\mathcal{M} = \mathrm{CM}(G, \Omega, p)$ is

- *balanced* if $p(x^{-1}) = (p(x))^{-1}$ (1992 Širáň, Škoviera),

- *anti-balanced* if $p(x^{-1}) = (p^{-1}(x))^{-1}$ (1994 Škoviera, Širáň),

- t*-balanced* if $p(x^{-1}) = (p^{t}(x))^{-1}$ (1998 Martino, Schultz).

Necessarily, $t^2 \equiv 1 \pmod{|\Omega|}$.

Note: balanced $\Leftrightarrow$ 1-balanced,

antibalanced $\Leftrightarrow$ -1-balanced.

Groups with sign structure

Širáň, Škoviera 1992:

• *group with sign structure*: group G with a given subgroup G^+ of index at most 2,

Let G be a group with a generating set Ω.
• *sign structure induced by* Ω:

$$G^+ = \{g \in G;\ g = y_1 \dots y_{2n},\ y_i \in \Omega,\ n \in \mathbb{N}_0\}$$

• *trivial* sign structure: $G = G^+$.

Theorem. *Let $\mathcal{M} = CM(G, \Omega, p)$ be a regular t-balanced Cayley map. Then either Ω induces a nontrivial sign structure on G or $\mathcal{M}$ is balanced.*

Trivial sign structure

It can be shown, that canonical double cover of regular balanced Cayley map with trivial sign structure is regular balanced Cayley map with *nontrivial* sign structure.

Therefore, we can assume that sign structures induced by our maps are nontrivial. Equivalently, underlying graph is *bipartite*.

If $[G : G^+] = 2$ we say that G is a $\mathbb{Z}_2$-*extension* of G^+.

t-automorphisms

Martino, Schultz, Škoviera :
Let $G = G^+(\theta, s)$ be a group with sign structure and $t \in \mathbb{Z} \setminus \{0\}$.

- t-*automorphism*: $\varphi \in \mathrm{Sym}(G)$, $\varphi(x) \in G^+ \Leftrightarrow x \in G^+$ and

$$\varphi(xy) = \varphi(x)\varphi^{\pi(x)}(y),$$

($\pi : G \to \{1, t\}$, $\pi(x) = 1$, for $x \in G^+$, $\pi(x) = t$, for $x \notin G^+$).

Note: $$t^2 \equiv 1 \pmod{\mathrm{ord}(\varphi)}.$$

Theorem. *Let $\mathcal{M} = CM(G, \Omega, p)$ be a Cayley map such that Ω induces a nontrivial sign structure on G. Then $\mathcal{M}$ is regular and t-balanced if and only if there exists a t-automorphism φ of G whose restriction to Ω is equal to p.*

φ is called the rotary t-automorphism of $\mathcal{M}$.

Regular t-balanced Cayley maps

Regular t-balanced Cayley map $\mathcal{M}$ is fully characterized by a triple (G, φ, x) where

- G is a group with sign structure,
- φ is a t-automorphism of G, and
- $x \in G$ is a member of a generating self-inverse orbit of φ.

New point of view

- underlying graph is bipartite,

- t is fixed, not necessarily $1 \leq t < |\Omega|$, i.e., 6-valent balanced map is also 7-balanced, -5-balanced, etc.

- basic object is G^+, instead of G.

  We say that $\mathcal{M} = CM(G, \Omega, p)$ is **from** G and **over** G^+.

G^+ as basic object: $\mathbb{Z}_2$-extensions

Širáň, Škoviera 1992:

Proposition. *Let G^+ be a group, $\theta \in \mathrm{Aut}(G^+)$, $s \in G^+$ be such that*

$$\begin{aligned} \theta(s) &= s \\ \theta^2 &= \nu_s \end{aligned}$$

($\nu_s(g) = sgs^{-1}$). Then the set $G^+ \times \mathbb{Z}_2$ together with the binary operation $\cdot$ defined by

$$(g,i)(h,j) = (g\theta^i(h)s^{ij}, i+j)$$

is a group denoted by $G^+(\theta, s)$. Moreover any $\mathbb{Z}_2$-extension of G^+ is isomorphic to some $G^+(\theta, s)$.

G^+ as basic object: t-automorphisms

Líšková, M, Škoviera 2007:

Let $G = G^+(\theta, s)$ be a group with sign structure. Let φ be a t-automorphism of G, $\xi = \varphi|_{G^+}$ and $w = \varphi(e,1)(e,1)^{-1}$ ($e = e_{G^+}$). Then $\xi \in \mathrm{Aut}(G^+)$, $w \in G^+$ and

$$\varphi(g,i) = (\xi(g)w^i, i).$$

We will write $\varphi \sim (\xi, w)$

Theorem. *Let $G = G^+(\theta, s)$ be a group with sign structure. Let $\xi \in \mathrm{Aut}(G^+)$, $w \in G^+$ be given, and put $\overline{w} = \xi^{-t}(\xi^{t-1}(w)\xi^{t-2}(w)\ldots\xi(w)w)$. Then the mapping $\varphi : (h,i) \mapsto (\xi(h)w^i, i)$ is a t-automorphism of G iff:*

$$\xi\theta(\overline{w}^{-1}s) = ws$$
$$\xi\theta = \nu_w\theta\xi^t = \nu_{ws}\theta^{-1}\xi^t.$$

G^+ as basic object: regular t-balanced Cayley maps

Regular t-balanced Cayley map $\mathcal{M}$ is fully characterized by a sextuple $(G^+, \theta, s, \xi, w, g)$ where $\theta, \xi \in \mathrm{Aut}(G^+)$, $s, w, g \in G^+$ and

- $G = G^+(\theta, s)$ is a group with sign structure,
- mapping $\varphi \sim (\xi, w)$ is a t-automorphism of G, and
- $(g, 1)$ is a member of a generating self-inverse orbit of φ.

t-algebras

Universal algebra $\mathcal{A} = (G^+, \circ, e, ^{-1}, \theta, s, \xi, w, g)$ of type $(2, 1, 0, 1, 0, 1, 0, 0)$ is a *t-algebra* if

- it satisfies all equations from the last three slides (they depend of t), and

- ξ is a permutation on G^+,

i.e, $(G^+, \circ, e, ^{-1})$ is a group, $G = G^+(\theta, s)$ is a group with sign structure and $\varphi \sim (\xi, w)$ is a t-automorphism of G.

If $x = (g, 1)$ is a member of a generating self inverse orbit of φ, we say that $\mathcal{A}$ *represents* the corresponding regular t-balanced Cayley map $\mathcal{M}$.

Product of regular t-balanced Cayley maps

Theorem. *Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a direct product of t-algebras. If $\mathcal{A}$ represents a regular t-balanced Cayley map $\mathcal{M}$, then there exist regular t-balanced Cayley maps $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{A}_1$ represents $\mathcal{M}_1$ and $\mathcal{A}_2$ represents $\mathcal{M}_2$, respectively.*

We say that $\mathcal{M}$ is a *product* of $\mathcal{M}_1$ and $\mathcal{M}_2$ and write

$$\mathcal{M} = \mathcal{M}_1 \odot \mathcal{M}_2$$

(this is a special case of the parallel product (Wilson 1994, Orbanic 2007))

Decomposition results I

Proposition. *The direct product decompositions of a t-algebra $\mathcal{A} = (G^+, \circ, e, ^{-1}, \theta, s, \xi, w, g)$ are in one to one correspondence with the inner product decompositions of the group G^+ into factors which are both θ- and ξ-invariant*

Theorem. *Every regular t-balanced Cayley map*

from above	*a/an*	cyclic Abelian nilpotent	*group*

is a product of regular t-balanced Cayley maps

"from" above	cyclic Abelian (nilpotent)	*p-groups.*

(If $2 \nmid |H|$, then "from" $H \Leftrightarrow$ from $H \times \mathbb{Z}_2$.)

Abelian groups

Lemma. *Let G^+ be an Abelian group of odd order (in the additive notation). Then every regular t-balanced Cayley map $\mathcal{M}$ from $G = G^+ \times \mathbb{Z}_2$ can be represented by a t-algebra $\mathcal{A}$ such that*
1) $\theta = \mathrm{id}$ *and* $s = 0$,
2) $\xi^{t-1} = \mathrm{id}$,
3) $(t+1) \times w = 0$ *and*
4) $\xi(w) = w$.

Lemma. *Let G^+ be a finite Abelian group and let $\psi \in \mathrm{End}(G^+)$. Then there exists a positive integer n such that*

$$G^+ = \mathrm{Ker}\,\psi^n \oplus \mathrm{Im}\,\psi^n.$$

Abelian p-groups, p-odd

Lemma. *Let G^+ be an Abelian p-group of odd order, let $\mathcal{A} = (G^+, +, 0, -, \mathrm{id}, 0, \xi, w, g)$ be a t-algebra representing regular t-balanced Cayley map $\mathcal{M}$ and let $\psi = \xi - \mathrm{id}$. Then, either $\mathcal{M}$ is balanced or*

1) $G^+ = \mathrm{Ker}\,\psi \oplus \mathrm{Im}\,\psi$,

2) $\mathrm{Ker}\,\psi = [w]$,

3) $\mathcal{M} = \mathcal{M}_a \odot \mathcal{M}_b$, where $\mathcal{M}_a$ is over $[w]$ and $\mathcal{M}_b$ is over $\mathrm{Im}\,\psi$,

4) $\mathcal{M}_a$ is antibalanced, and

5) $\mathcal{M}_b$ is balanced.

Both $\mathcal{M}_a$ and $\mathcal{M}_b$ are t-balanced, too, i.e., the valency of $\mathcal{M}_a$ divides $t+1$ and the valency of $\mathcal{M}_b$ divides $t-1$.

Decomposition results II

Theorem. *Every regular t-balanced Cayley map $\mathcal{M}$ from an Abelian group can be expressed as a product*

$$\mathcal{M} = \mathcal{M}_a \odot \mathcal{M}_b \odot \mathcal{M}_2,$$

where

- $\mathcal{M}_a$ *is antibalanced from* $\mathbb{Z}_n \times \mathbb{Z}_2$,
- $\mathcal{M}_b$ *is balanced from* $H \times \mathbb{Z}_2$, $|H|$ *is odd, and*
- $\mathcal{M}_2$ *is from a 2-group.*

(We allow some factors to be trivial, i.e., K_2.)

Abelian 2-groups?

Proposition. *Any regular t-balanced Cayley map $\mathcal{M}$ from $\mathbb{Z}_{2^{e+1}}$ is antibalanced. Moreover, the valency of $\mathcal{M}$ is one of 2^e ($e > 2$ two embeddings), 4 and 2.*

Proposition. *Any regular t-balanced Cayley map $\mathcal{M}$ from $\mathbb{Z}_{2^e} \times Z_2$ with $G^+ = \mathbb{Z}_{2^e}$ has valency 2^e. Moreover it is antibalanced or $2^{e-1} - 1$-balanced.*

Thank You