JOURNAL OF COMBINATORIAL THEORY, Series B 37, 221–227 (1984)

New Sufficient Conditions for Cycles in Graphs

GENG-HUA FAN*

Department of Mathematics, University of Fuzhou, Fuzhou, Fujian, People's Republic of China

Communicated by the Managing Editors

Received January 25, 1984

A new sufficient condition for a graph to be Hamiltonian is given that does not require that the closure of the graph should be complete, and so it is independent of the conditions given by Bondy (*Discrete Math.* 1 (1971), 121–132) and Chvátal (*J. Combin. Theory Ser. B* 12 (1972), 163–168). For the circumference, length of a longest cycle, a similar result is also obtained. The theorems to be presented in this paper are generalizations of the theorem of Ore (*Amer. Math. Monthly* 67 (1960), 55) in a new direction.

1. INTRODUCTION

All graphs considered in this paper are undirected and have no loops or multiple edges. $V(G)$ and $E(G)$ denote the sets of vertices and edges of the graph G, respectively. The degree of vertex v is denoted by $d(v)$, and d_i represents the ith term of the degree sequence of G $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$, where $n = |V(G)|$. (v, u) denotes the edge between vertices v and u, and the distance between v and u, denoted by $d(v, u)$, is the minimum length of a v–u path.

The closure of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices, whose degree sum is at least $|V(G)|$, until no such pair remains.

The following results are known: A graph G on $n \geqslant 3$ vertices is hamiltonian if:

(1.1) Dirac [1]. $d(v) \geqslant n/2$ for every vertex v in G.

(1.2) Ore [2]. $(v, u) \bar{\in} E(G) \Rightarrow d(v) + d(u) \geqslant n$.

(1.3) Pósa [3]. $|\{v: d(v) \leqslant j\}| < j$ for $j < (n-1)/2$ and $|\{v: d(v) \leqslant (n-1)/2\}| \leqslant (n-1)/2$ for n odd.

(1.4) Bondy [4]. $j < k$, $d_j \leqslant j$, $d_k \leqslant k-1 \Rightarrow d_j + d_k \geqslant n$.

(1.5) Chvátal [5]. $d_j \leqslant j < n/2 \Rightarrow d_{n-j} \geqslant n-j$.

* Present address: Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada.

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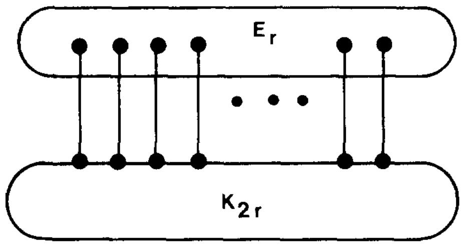

FIGURE 1

All theorems above share the same property: if a graph satisfies the condition given there then its closure must be complete. This property is so strong that many hamiltonian graphs are excluded. Consider the graph containing a complete graph K_{2r} $(r \geqslant 2)$, a set of r independent edges, denoted by E_r, and a matching between the sets of vertices of K_{2r} and E_r (see Fig. 1). This graph has $4r$ vertices and its degree sequence is $(2, 2, \ldots, 2, 2r, \ldots, 2r)$ with $2r$ terms equal to 2 and $2r$ terms equal to $2r$. It can be easily seen that the graph fails to satisfy any one of the theorems above, since its closure is itself, far from being complete. It is a surprise that the theorem to be presented in this paper indeed ensures that the graph is hamiltonian.

2. HAMILTONIAN CYCLES

THEOREM 1. *Let G be a 2-connected graph on $n \geqslant 3$ vertices and let v and u be distinct vertices of G. If*

$$d(v, u) = 2 \Rightarrow \max(d(v), d(u)) \geqslant n/2$$

then G has a hamiltonian cycle.

Remark. The requirement of 2-connectedness is necessary. Any sufficient condition for hamiltonian cycles must contains this. Note that $(v, u) \overline{\in} E(G)$ implies that $d(v, u) \geqslant 2$ so it is clear that Theorem 1 is much stronger than Ore's (1.2) in the Introduction.

Proof. Suppose that G is a graph satisfying the given condition and G has no hamiltonian cycle. We shall arrive at a contradiction.

Let $P = v_0 v_1 \cdots v_m$ be a longest path in G of length m, chosen so that $d(v_0) + d(v_m)$ is as large as possible. If $d(v_0) + d(v_m) \geqslant n$ then there are at least two consecutive vertices on P, v_l and v_{l+1} such that $(v_l, v_m) \in E(G)$ and $(v_{l+1}, v_0) \in E(G)$, and so we obtain a cycle of length $m+1$. By the connectedness of G we have either a hamiltonian cycle or a path of length $m+1$. Both lead to contradictions. Consequently $d(v_0) + d(v_m) < n$. We can suppose without loss of generality that

$$d(v_0) < n/2.$$

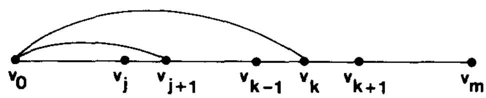

FIGURE 2

From the proof above we can also suppose that

(a) G has no cycle of length $m+1$.

Since G is 2-connected, v_0 is joined to at least one vertex on P other than v_1. Choose v_k for which $(v_0, v_k) \in E(G)$ and k is as large as possible (see Fig. 2). Clearly $k \geqslant 2$, and $k < m$, otherwise we have a cycle of length $m+1$, which is contrary to (a). We show now that

(b) $(v_0, v_i) \in E(G)$ for all i, $1 \leqslant i \leqslant k$.

(c) $d(v_i) \leqslant d(v_0)$ for all i, $1 \leqslant i \leqslant k-1$.

(d) $d(v_{k+1}) > d(v_0)$.

Proof. (b) As $(v_0, v_k) \in E(G)$, there is another longest path $v_{k-1}v_{k-2} \cdots v_0 v_k \cdots v_m$. By the maximality of $d(v_0)+d(v_m)$, we have that $d(v_{k-1}) \leqslant d(v_0) < n/2$. Then $\max(d(v_{k-1}), d(v_0)) < n/2$. It follows from the condition of the theorem that $d(v_0, v_{k-1}) \neq 2$. However, $v_0 v_k v_{k-1}$ is a path of length 2 and thus we must have that $(v_0, v_{k-1}) \in E(G)$. If $k-1=1$ we stop, if not we repeat the process above and thus we obtain that $(v_0, v_i) \in E(G)$ for all i, $1 \leqslant i \leqslant k$.

(c) Suppose that $d(v_j) > d(v_0)$ for some j, $1 \leqslant j \leqslant k-1$. Since $(v_0, v_{j+1}) \in E(G)$ by (b), $v_j v_{j-1} \cdots v_0 v_{j+1} \cdots v_k \cdots v_m$ is another longest path with $d(v_j)+d(v_m) > d(v_0)+d(v_m)$, which contradicts the maximality of $d(v_0)+d(v_m)$.

(d) Note that $(v_0, v_{k+1}) \bar{\in} E(G)$ by the choice of v_k and the path $v_0 v_k v_{k+1}$ is of length 2, we find that $d(v_0, v_{k+1}) = 2$. Using the condition of the theorem we obtain that $\max(d(v_0), d(v_{k+1})) \geqslant n/2$. But $d(v_0) < n/2$ and so $d(v_{k+1}) \geqslant n/2 > d(v_0)$.

Having shown the assertions above, we go now into the main part of the proof of Theorem 1.

Notice first that for every i, $1 \leqslant i \leqslant k-1$, v_i cannot be joined to any vertex outside P, since there is, by (b), a longest path $v_i v_{i-1} \cdots v_0 v_{i+1} \cdots v_k \cdots v_m$. The 2-connectedness of G, therefore, implies that there exists $(v_j, v_s) \in E(G)$ such that $j < k < s$. Choose

$$(v_j, v_s) \in E(G)$$

for which $j < k < s$ and s is as large as possible.

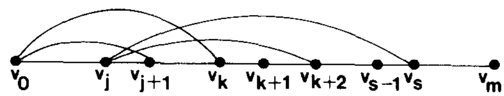

FIGURE 3

Case 1. $s \geqslant k+2$ (see Fig. 3). By (b), $(v_0, v_{j+1}) \in E(G)$ and so there is a longest path $v_{s-1}v_{s-2} \cdots v_{j+1}v_0 \cdots v_jv_s \cdots v_m$. It follows from the maximality of $d(v_0)+d(v_m)$ that $d(v_{s-1}) \leqslant d(v_0) < n/2$. On the other hand, by (c), $d(v_j) \leqslant d(v_0) < n/2$. Consequently $\max(d(v_{s-1}), d(v_j)) < n/2$. Using the condition of the theorem we have that $d(v_j, v_{s-1}) \neq 2$, but $v_jv_sv_{s-1}$ is a path of length 2 and thus we must have that $(v_j, v_{s-1}) \in E(G)$. If $s-1 > j+1$ we have another longest path $v_{s-2}v_{s-3} \cdots v_{j+1}v_0 \cdots v_jv_{s-1}v_s \cdots v_m$. Repeating the process we obtain that $(v_j, v_{s-2}) \in E(G)$. If $s-2 > j+1$ we go through the same process again. Consequently we have that

$$(v_j, v_i) \in E(G) \qquad \text{for all} \quad i, \qquad j+1 \leqslant i \leqslant s.$$

In particular, $(v_j, v_{k+2}) \in E(G)$ since $s \geqslant k+2$. This means that there is a longest path $v_{k+1}v_k \cdots v_{j+1}v_0 \cdots v_jv_{k+2} \cdots v_m$ with $d(v_{k+1})+d(v_m) > d(v_0)+d(v_m)$, by (d). This is impossible by the maximality of $d(v_0)+d(v_m)$.

Case 2. $s=k+1$ (see Fig. 4). Note first that $v_kv_{k-1} \cdots v_{j+1}v_0 \cdots v_jv_{k+1} \cdots v_m$ is a longest path and so by the maximality of $d(v_0)+d(v_m)$ we have that

$$d(v_k) \leqslant d(v_0) < n/2. \tag{1}$$

If $k+1=m$ we obtain a cycle $v_0v_1 \cdots v_jv_{k+1}v_k \cdots v_{j+1}v_0$ of length $m+1$. This contradicts (a). So we may assume that

$$k+1 < m.$$

It follows from the 2-connectedness of G and the choice of v_s that there must be

$$(v_k, v_t) \in E(G) \qquad \text{such that} \quad t \geqslant k+2.$$

This implies that there is another longest path $v_{t-1}v_{t-2} \cdots v_{k+1}v_j \cdots v_0v_{j+1} \cdots v_kv_t \cdots v_m$, and then by the maximality of $d(v_0)+d(v_m)$ we have that $d(v_{t-1}) \leqslant d(v_0) < n/2$. Together with (1) this implies that

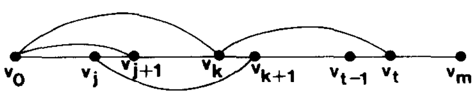

FIGURE 4

$\max(d(v_k), d(c_{t-1})) < n/2$, and so by the condition of the theorem $d(v_k, v_{t-1}) \neq 2$. Note that $v_k v_t v_{t-1}$ is a path of length 2 we obtain $(v_k, v_{t-1}) \in E(G)$. If $t-1 > k+1$ we repeat the process. By the same argument as before we have

$$(v_k, v_i) \in E(G) \qquad \text{for all } i, \qquad k+1 \leqslant i \leqslant t.$$

In particular, $(v_k, v_{k+2}) \in E(G)$ since $t \geqslant k+2$. Hence, there is a longest path $v_{k+1} v_j \cdots v_0 v_{j+1} \cdots v_k v_{k+2} \cdots v_t \cdots v_m$ with $d(v_{k+1}) + d(v_m) > d(v_0) + d(v_m)$, by (d). This is contrary to the maximality of $d(v_0) + d(v_m)$. This completes the proof of Theorem 1.

Applying Theorem 1 to the graph in Fig. 1, one is able to say that the graph has indeed a hamiltonian cycle. This fact shows that Theorem 1 is not weaker than any one of the theorems from (1.1) to (1.5). Unfortunately, it can be easily seen that Theorem 1 is not stronger than (1.3), (1.4), and (1.5), either. However, at the end of this paper a general theorem will be given that is stronger than all theorems mentioned above, with the exception of Chvátal's (1.5) in the Introduction.

3. The Circumference

The following are existing results: G is a 2-connected graph on n vertices and $3 \leqslant c \leqslant n$. If

(3.1) Dirac [1]. $d(v) \geqslant c/2$ for every $v \in V(G)$,

(3.2) Bermond [6]. $(v, u) \bar{\in} E(G) \Rightarrow d(v) + d(u) \geqslant c$,

(3.3) Pósa [7]. $|\{v: d(v) \leqslant j\}| \leqslant j-1$ for $1 \leqslant j \leqslant (c-1)/2$,

(3.4) Bondy [4]. $d_j \leqslant j,\ d_k \leqslant k (j \neq k) \Rightarrow d_j + d_k \geqslant c$,

then G has a cycle of length at least c.

In this section we need the following well-known result by Bondy [4].

Lemma 1. *Let G be a 2-connected graph on n vertices and $3 \leqslant c \leqslant n$. Suppose that P is a longest path in G with end vertices v and u. If $d(v) + d(u) \geqslant c$ then G contains a cycle of length at least c.*

Proof. A proof can be found in [4].

Based on the proof of Theorem 1 one can deduce

Theorem 2. *Let G be a 2-connected graph on n vertices and $3 \leqslant c \leqslant n$. If*

$$d(v, u) = 2 \Rightarrow \max(d(v), d(u)) \geqslant c/2$$

for each pair of vertices v and u in G, then G has a cycle of length at least c.

Proof. The proof is very similar to that of Theorem 1. By replacing $n/2$ there by $c/2$ here and using Lemma 1 one can easily complete the proof and thus we omit it.

Applying any one of the theorems from (3.1) to (3.4) to the graph in Fig. 1, one can only say that the graph has a cycle of length at least 4, while using Theorem 2 we have that the graph has a hamiltonian cycle.

It is not difficult to see that Theorem 2 is equivalent to

THEOREM 3. *Let G be a 2-connected graph on n vertices and* $3 \leqslant c \leqslant n$, *and let* $S = \{v: d(v) < c/2\}$. *If*

$$v, u \in S \Rightarrow d(v, u) \neq 2$$

then G has a cycle of length at least c.

4. A General Theorem

This section is devoted to

THEOREM 4. *Let G be a 2-connected graph on n vertices and* $3 \leqslant c \leqslant n$. *Let*

$$S_j = \{v: d(v) \leqslant j\} \qquad \text{for} \quad j = 2, 3, \ldots, n.$$

If

$$j < k, j + k < c + 1, |S_j| \geqslant j, \text{ and } |S_{k-1}| \geqslant k \Rightarrow d(v, u) \neq 2$$

for every $v, u \in S_j$, *then G has a cycle of length at least c.*

Proof. Let $P = v_0 v_1 \cdots v_m$ be a longest path in G such that $d(v_0) + d(v_m)$ is as large as possible. Without loss of generality, suppose that

$$j = d(v_0) \leqslant d(v_m) = k - 1.$$

Then $j < k$. If $j + (k-1) = d(v_0) + d(v_m) \geqslant c$ then the proof is completed by Lemma 1. So we have that $j + k < c + 1$. Put

$$N'(v_0) = \{v_i: (v_0, v_{i+1}) \in E(G)\}$$

and

$$N'(v_m) = \{v_i: (v_m, v_{i-1}) \in E(G)\}.$$

For $v_i \in N'(v_0)$, since $v_i v_{i-1} \cdots v_0 v_{i+1} v_{i+2} \cdots v_m$ is another longest path, the maximality of $d(v_0) + d(v_m)$ gives that $d(v_i) \leqslant d(v_0) = j$. Similarly

$d(v_i) \leqslant d(v_m) = k-1$ for $v_i \in N'(v_m)$. So we have $N'(v_0) \subseteq S_j$ and $N'(v_m) \subseteq S_{k-1}$, and thus $|S_j| \leqslant |N'(v_0)| = j$ and $|S_{k-1}| \geqslant |N'(v_m)| = k-1$. Note that $v_0 \in N'(v_m)$ and $d(v_0) \leqslant d(v_m) = k-1$ we see that $|s_{k-1}| \geqslant k$. This implies that we can use the given condition that

$$d(v, u) \neq 2 \qquad \text{for every} \quad v, u \in S_j.$$

However, this is just the base of the whole proof of Theorem 1. By the same discussion as used there, one can easily complete the remainder of the proof and thus we omit it.

We show now that Theorem 4 is stronger than all theorems appearing in the preceding sections, with the exception of Chvátal's (1.5). In order to do this, it clearly suffices to show that if G satisfies the conditions of Bondy's (3.4) then G must satisfy the condition of Theorem 4. Suppose that G satisfies the condition of (3.4). If G fails to satisfy the condition of Theorem 4 then there exist some j and k such that $j < k$, $j + k < c + 1$, $|S_j| \geqslant j$, and $|S_{k-1}| \geqslant k$. It is easy to see that the last two inequalities imply that $d_j \leqslant j$ and $d_k \leqslant k-1$.

If $k > j+1$ then $d_{k-1} \leqslant d_k \leqslant k-1$, and then

$$d_j + d_{k-1} \leqslant j + (k-1) < c.$$

If $k = j+1$ then $d_{j+1} = d_k \leqslant k - 1 = j < j+1$, and then

$$d_j + d_{j+1} \leqslant 2j < c.$$

Both contradict the supposition that G satisfies the condition of (3.4).

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