Graph Theory
HOMEWORK #1.
Deadline: October 27, midnight

Basic rules for submitting your homework:

- *the file has to be named: "firstname-lastname-HW1.pdf"*
- *put your name on the top of each page*
- *number the pages*
- *clearly mark which problem is being solved and list the solutions in the order of the problems from the assignment*
- *write the solutions carefully providing sufficient details; when in doubt, write more*
- *HW is to be turned in via* moodle *in the* pdf *format*
- *late homeworks only accepted in justified situations*

The total number of points in this HW is greater than 50 points. You cannot receive more than 50 points, so you may choose to just solve problems worth 50 pts. However, if you choose to do that, and some of your solutions turn out wrong or imperfect, you will loose points and will not receive the full 50 pts. On the other hand, if you choose to solve problems worth more than 50 points, even with some mistakes in some of the problems you may get enough done to earn the maximum of 50 points.
Your solutions do not have to be perfect to receive full credit, but you should try to make your solutions as close to perfect as you can without spending your life on the homework.

1. (8 pts) Eulerian Circuit

   Prove or disprove that every finite digraph (each edge is directed) with the property that the in-degree of each of its vertices is equal to its out-degree admits a directed Eulerian circuit.

2. (8 pts) Eulerian Circuit

   Suppose a graph Γ is not eulerian. Design an algorithm that will find a closed circuit in Γ that will contain the maximal number of edges among all closed circuits in Γ. Derive a meaningful estimate for the number of edges contained in the maximal closed circuit in Γ, and prove your claims.

3. (8 pts) Tournaments

   An oriented complete graph graph is called a *tournament*. Show that every tournament contains a directed Hamilton path. Is the same true for a directed Hamilton circuit?

4. (8 pts) Hamiltonicity of Circulants

   A graph of order n is called a *circulant* if it admits an automorphism consisting of a single cycle of length n, i.e., if it admits an automorphism φ mapping the vertices in the 'circular manner':

   $$u_1 \mapsto u_2 \mapsto u_3 \mapsto \ldots \mapsto u_n \mapsto u_1$$

   a) Prove or disprove that every circulant is regular, i.e., all of its vertices are of the same degree.

   b) Prove or disprove that a circulant is connected if and only if it is k-regular with $k \geq 2$.

   c) Prove or disprove that every connected circulant is eulerian.

   d) Prove or disprove that every connected circulant is hamiltonian.

5. (8 pts) Sabidusi's Conjecture for Circulants

   Prove that every k-regular circulant with even $k \geq 4$ admits for every Eulerian circuit $\mathcal{C}$ a partition of its edge set into cycles having the property that none of them contains two consecutive edges that are also consecutive in $\mathcal{C}$, i.e., prove that the Sabidusi conjecture holds for k-regular circulants with even $k \geq 4$. Does Sabidusi's conjecture hold for 2-regular circulants?

6. (8 pts) Necessary Condition for Hamiltonicity

   Find an infinite family of non-hamiltonian graphs G that satisfy the condition that for every non-empty subset $S \subseteq V(G)$ the number of components of $G \setminus S$ is at most the cardinality of S.

7. (8 pts) Non-Hamiltonian Sequence

   Find a hamiltonian graph whose degree sequence is non-hamiltonian.

8. (8 pts) Square of a Graph

   Find a connected graph G whose square G^2 is non-hamiltonian.

9. (8 pts) Hamilton Path vs. Hamilton Cycle

   Find or prove the non-existence of a 4-regular graph that admits a Hamilton path but not a Hamilton cycle.