Graph Theory

HOMEWORK #2.

Deadline: Thursday, November 21, midnight

Basic rules for submitting your homework:

- *the file has to be named: "firstname-lastname-HW2.pdf"*
- *put your name on the top of each page*
- *number the pages*
- *clearly mark which problem is being solved and list the solutions in the order of the problems from the assignment*
- *write the solutions carefully providing sufficient details; when in doubt, write more*
- *HW is to be turned in via* moodle *in the* pdf *format*
- *late homeworks only accepted in justified situations*

The total number of points in this HW is greater than 50 points. You cannot receive more than 50 points, so you may choose to just solve problems worth 50 pts. However, if you choose to do that, and some of your solutions turn out wrong or imperfect, you will loose points and will not receive the full 50 pts. On the other hand, if you choose to solve problems worth more than 50 points, even with some mistakes in some of the problems you may get enough done to earn the maximum of 50 points.

Your solutions do not have to be perfect to receive full credit, but you should try to make your solutions as close to perfect as you can without spending your life doing the homework.

1. (25 pts) Hamiltonian and Hamiltonian-connected graphs

   a) Prove that if a graph Γ is such that each of its vertices is of even degree, then the line graph of Γ, $L(\Gamma)$, contains a Hamiltonian cycle.

   b) Determine whether the line graph $L(\Gamma)$ of a graph Γ in which each of its vertices is of even degree must be Hamilton-connected. A graph Γ is **Hamilton-connected** if for every pair of vertices $u, v \in V(\Gamma)$ there exists a Hamiltonian path from u to v. Justify your answer.

   c) Prove that the line graph of a Hamiltonian graph is Hamiltonian.

   d) Let Γ be a graph with fewer than i vertices of degree at most i for every $i < |V(\Gamma)|/2$. Show that Γ is Hamiltonian.

   e) Use induction on the order of a graph Γ to show that the third power Γ^3 of a connected graph is Hamilton-connected.

2. (5 pts) Write a quick survey on what is known about Hamilton-connected graphs (a few necessary and sufficient conditions). Use any source you wish.

3. (10 pts) Hamiltonicity of Circulants

   If you did not do this problem in our first homework, or if you solved it in an 'easy way' by considering an extreme (small or very special) case, try one more time to solve the following exercise in more generality.

   A graph of order $n \geq 3$ is called a *circulant* if it admits an automorphism consisting of a single cycle of length n, i.e., if it admits an automorphism φ mapping the vertices in the 'circular manner':

$$u_1 \mapsto u_2 \mapsto u_3 \mapsto \ldots \mapsto u_n \mapsto u_1$$

   a) Prove or disprove that every circulant is regular, i.e., all of its vertices are of the same degree.

   b) Prove or disprove that a circulant is connected if and only if it is k-regular with $k \geq 2$.

   c) Prove or disprove that every connected circulant is eulerian.

   d) Prove or disprove that every connected circulant is hamiltonian.

4. (15 pts)

a) Let Γ be a graph of order at least n, and let Γ' be a graph of order less than n. Determine the automorphism group of the graph constructed from disjoint copies of the graphs Γ, Γ' by connecting every vertex of Γ' to all the vertices of Γ. Justify your answer.

b) Can you say anything meaningful about the automorphism groups of graphs constructed from disjoint copies of the graphs Γ, Γ' by connecting every vertex of Γ' to exactly one vertex of Γ? If your answer is 'chaos', meaning that the resulting graphs might end up with disparate automorphism groups, justify your answer with some examples. This sort of answer may also be used in part a), if you think that is the case.

5. (5 pts) Find a 3-regular graph Γ of order bigger than 4 that has the property that all graphs obtained from Γ by removing any single edge are isomorphic. Justify your answer. Note that K_4 does have this property.

6. (5 pts) Prove or disprove the existence of a connected vertex-transitive graph Γ that is not isomorphic to a complete graph K_n, for any $n \geq 1$, and which has the property that $\Gamma \setminus \{u\}$ is vertex-transitive for each $u \in V(\Gamma)$ (u is removed together with all of its adjacent vertices). Justify your answer. Note that K_n does have this property for all $n \geq 1$.