Graph Theory

HOMEWORK #3.

Deadline: December 19, 2024, midnight

Basic rules for submitting your homework:

- *the file has to be named: "firstname-lastname-HW3.pdf"*
- *put your name on the top of each page*
- *number the pages*
- *clearly mark which problem is being solved and list the solutions in the order of the problems from the assignment*
- *write the solutions carefully providing sufficient details; when in doubt, write more*
- *HW is to be turned in via* moodle *in the* pdf *format*
- *late homeworks only accepted in justified situations*

The total number of points in this HW is greater than 50 points. You cannot receive more than 50 points, so you may choose to just solve problems worth 50 pts. However, if you choose to do that, and some of your solutions turn out wrong or imperfect, you will loose points and will not receive the full 50 pts. On the other hand, if you choose to solve problems worth more than 50 points, even with some mistakes in some of the problems you may get enough done to earn the maximum of 50 points.

Your solutions do not have to be perfect to receive full credit, but you should try to make your solutions as close to perfect as you can without spending your life doing the homework.

1. (5pts) Find a vertex-transitive graphs that is not strongly regular. Justify your answer.

2. (30 pts) Graphs with trivial automorphism group

   a) Prove or disprove that the disjoint union of asymmetric graphs is asymmetric.

   b) Prove or disprove that the graph that is obtained from a disjoint union of non-isomorphic asymmetric graphs by adding edges between all the vertices of the first graph and all the vertices of the second graph is asymmetric.

   c) Determine whether the line graph $L(\Gamma)$ of an asymmetric graph Γ is necessarily asymmetric.

   d) Determine whether the square Γ^2 of an asymmetric graph Γ is necessarily asymmetric.

   f) Does there exist an asymmetric graph Γ of diameter $d \geq 2$ for which the powers Γ^i are asymmetric for all $2 \leq i \leq d$?

3. (10pts) Trivial stabilizer

   Let Γ be a graph, and let $G = Aut(\Gamma)$ be its automorphism group having the property that $Stab_G(v) = \langle 1_G \rangle$, for all $v \in V(\Gamma)$. Prove that all orbits of G on the vertices of Γ are of the same size (such an action is called *semi-regular*).

4. (10pts) Edge-transitivity

   a) Find a graph Γ that is vertex-transitive but not edge-transitive.

   b) Find a graph Γ that is edge-transitive but not vertex-transitive.

5. (10 pts) Paley graphs

   a) Prove that Paley graphs are strongly regular.

   b) Does there exist a Paley graph that has a trivial stabilizer?

6. (20 pts) Circulants

   a) Find at least three circulants that are strongly regular graphs.

   b) Find at least three circulants that are self-complementary (i.e., isomorphic to their complement).

c) Prove or disprove that every circulant of order ≥ 3 has a non-trivial stabilizer.

d) Is there an upper bound that is greater than the ratio $\frac{|Aut(C(\mathbb{Z}_n,S))|}{n}$ for all circulants $C(\mathbb{Z}_n, S)$?

7. (30 pts) Cayley graphs

a) Prove that a Cayley graph $C(G, X)$ is connected if and only if X generates G.

b) Prove that a Cayley graph $C(G, X)$ of an abelian group of order ≥ 3 must have a non-trivial vertex stabilizer.

c) Let G be a group that contains a subgroup K of index 2. What can you say about the Cayley graph $C(G, G - K)$? Is it connected? Is it bipartite? Is it necessarily self-complementary (i.e., isomorphic to its complement)?

d) Let $\mathbb{S}_n$ be the full symmetric group and X be the set of 2-cycles $(i, j) \in \mathbb{S}_n$. What can you say about the Cayley graph $C(\mathbb{S}_n, X)$? What is its degree? What is the order of a vertex-stabilizer? Is it connected? Is it bipartite? Can you find a meaningful bound on its diameter?

e) Prove or disprove the statement: *If two Cayley graphs $C(G, X)$ and $C(G', X')$ are isomorphic, then the groups G and G' are isomorphic.*

f) Find a group G with two generating sets $X^1 = \{x_1^1, x_2^1, \ldots, x_k^1\}$, $X^2 = \{x_1^2, x_2^2, \ldots, x_k^2\}$ satisfying the property that the order of x_i^1 is equal to the order of x_i^2, for every $1 \leq i \leq k$, and such that the two Cayley graphs $C(G, X^1)$ and $C(G, X^2)$ are not isomorphic.