Graph Theory Lecture #10 Highly symmetric graphs

November 28, 2024

The Stabilizer of a Vertex in a Group of Permutations

Definition

Let $G \leq \mathbb{S}_X$, and $x \in X$. The **stabilizer of** x in G is the set of all permutations in G that fix (stabilize) x; i.e.,

$$Stab_G(x) = \{ \varphi \in G \mid \varphi(x) = x \}$$

- ⋆ $Stab_G(x)$ is a subgroup of G for every x ∈ X
- * if there exists a permutation $\varphi \in G$ such that $\varphi(x) = y$, for some $x, y \in X$, then $|Stab_G(x)| = |Stab_G(y)|$, and $Stab_G(x) = \varphi Stab_G(y) \varphi^{-1}$
- * The proof follows from the easy observations $Stab_G(x) \le \varphi Stab_G(y)\varphi^{-1}$ and $\varphi Stab_G(y)\varphi^{-1} \le Stab_G(x)$

The Stabilizer of a Vertex in a Graph

Definition

Let $\Gamma = (V, E)$ be a graph, $G = Aut(\Gamma)$, and $u \in V$. The **stabilizer of** u in G is the set of all automorphisms of Γ that fix (stabilize) u; i.e.,

$$Stab_G(u) = \{ \varphi \in G \mid \varphi(u) = u \}$$

- * all of these concepts are 'specifications' of the general results on groups of permutations to groups of permutations that are also graph automorphisms
- ⋆ $Stab_G(u)$ is a subgroup of G for every u ∈ V
- * if there exists an automorphism $\varphi \in G$ such that $\varphi(u) = v$, for some $u, v \in V$, then $|Stab_G(u)| = |Stab_G(v)|$, and $Stab_G(u) = \varphi Stab_G(v) \varphi^{-1}$

The Stabilizer of a Set

Definition

1. Let $\Gamma = (V, E)$ be a graph, $G = Aut(\Gamma)$, and $S \subseteq V$. The **pointwise stabilizer of** S in G is the set of all automorphisms of Γ that fix (stabilize) every $u \in S$; i.e.,

$$Stab_G(S) = \{ \varphi \in G \mid \varphi(u) = u, \text{ for all } u \in S \}$$

2. The **setwise stabilizer of** S in G is the set of all automorphisms of Γ that fix (stabilize) the set S (but may mix the elements in S); i.e.,

$$Stab_G(\{S\}) = \{ \varphi \in G \mid \varphi(u) \in S, \text{ for all } u \in S \}$$

- * $Stab_G(S) \leq Stab_G(\{S\}) \leq G$, for every $S \subseteq V$
- $\star Stab_G(S) = \bigcap_{u \in S} Stab_G(u)$
- * all the above definitions and observations clearly extend to general groups of permutations $G \leq \mathbb{S}_X$

Orbit of a Permutation Group

- * Recall the cycle decomposition of a permutation $\varphi \in \mathbb{S}_n$ into disjoint cycles.
- * Note that the cycles (including the single element cycles which we usually do not list) form a partition of the set $\{1, 2, ..., n\}$
- \star also note that any two elements i and j in $\{1, 2, ..., n\}$ belong to the same cycle if and only if there exists a $0 \le k \le |\varphi| - 1$ such that $\varphi^k(i) = i$

These observations generalize into the following:

Definition

Let G be a subgroup of the full symmetric group S_X . The **orbit** of G containing the element $x \in X$ is the set of all elements $y \in X$ such that there exists and element $\varphi \in G$ such that $\varphi(x) = y$.

Orbits of a Permutation Group

Theorem

- 1. If $x, y \in X$ belong to the same orbit of $G \leq \mathbb{S}_X$ on X, their stabilizers are conjugates in G^1 .
- 2. If $G \leq S_X$, the orbits of G on X form a partition of X.
- 3. The relation on X defined for any pair of elements $x, y \in X$ via the rule

$$x \sim y$$
 iff $\exists \varphi \in G$ such that $\varphi(x) = y$

is an equivalence relation on X

4. The cycles of φ are the orbits of $\langle \varphi \rangle$.

¹Is this an if and only if statement?

Lemma

Let $G \leq S_X$ and let \mathcal{O} be an orbit of G. Then,

$$|G| = |Stab_G(x)| \cdot |\mathcal{O}|,$$

for every $x \in \mathcal{O}$.

Proof: Let $\mathcal{O} = \{x = x_1, x_2, \dots, x_{|\mathcal{O}|}\}$ and $\varphi_1, \varphi_2, \dots, \varphi_{|\mathcal{O}|} \in G$ such that $\varphi_i(x_1) = x_i$. Then

$$G = \bigcup_{1 < i < |\mathcal{O}|} \varphi_i Stab_G(x_1)$$

Orbit of an Automorphism Group of a Graph

- \star Orbits with respect to the automorphism group of a graph Γ form a partition of the vertex set $V(\Gamma)$
- \star For any two vertices $u, v \in G(\Gamma)$ that belong to the same orbit, there exists an automorphism $\varphi \in Aut(\Gamma)$ mapping u to $v, \varphi(u) = v$
- \star If u and w in $V(\Gamma)$ do not belong to the same orbit, there is **no** automorphism $\varphi \in Aut(\Gamma)$ mapping u to v
- * a graph Γ has a trivial automorphism group if and only each vertex of Γ is its own orbit, i.e., if Γ has a non-trivial automorphism group, it has at least one orbit of size > 1.
- * What are the orbits of K_n , $K_{m,n}$, C_n and a star?

Vertex-Transitive Graphs

Definition

A graph $\Gamma = (V, E)$ is called **vertex-transitive** if for every pair $u, v \in V$ there exists an automorphism $\varphi \in Aut(\Gamma)$ such that $\varphi(u) = v$.

I.e., a graph $\Gamma = (V, E)$ is vertex-transitive if and only if $Aut(\Gamma)$ has exactly one orbit on $V(\Gamma)$.

Examples of vertex-transitive graphs:

- \star a vertex-transitive graph of order n is necessarily k-regular for some 0 < k < n - 1
- \star complete graphs K_n , n > 2
- * the Petersen graph
- * Johnson graphs J(n, k)
- * Kneser graphs K(n,k)



Vertex-Transitive Graphs

Problem: Show that the Johnson and Kneser graphs are vertex-transitive.

- * Is a strongly regular graph necessarily vertex-transitive?
- * Is a Latin square graph necessarily vertex-transitive?
- * Is a Latin square graph whose Latin square is a multiplication table for a group necessarily vertex-transitive?
- \star Is the group of automorphisms of a group G necessarily transitive on the elements of G?

Transitive permutation groups

Definition

Recall that a group $G \leq \mathbb{S}_X$ is **transitive (acts transitively) on** X, if for every pair of vertices $x, y \in X$ there exists a permutation φ , such that $\varphi(x) = y$.

Examples:

- \star the full symmetric group \mathbb{S}_X acts transitively on X
- \star the alternating group \mathbb{A}_X acts transitively on X

More examples of transitive permutation groups

- \star if G is a group, let $G_L = \{ \sigma_g \mid g \in G \}$, where $\sigma_g(h) = gh$, for all $h \in G$
- $\star G_L \leq \mathbb{S}_G$ and G_L acts transitively on G;
- ▶ this is called the (left) regular representation of G

More examples of transitive permutation groups

* Let G be a group, H be a subgroup of G, and let X be the set of left cosets of H in G

$$X = \{1_G H = g_1 H, g_2 H, g_n H\},\$$

where n = [G : H]

- \star for every element $g \in G$ define a permutation $\sigma_{g,H}$ of the set X by the rule $\sigma_{g,H}(g_iH) = gg_iH$; is it a permutation?
- \star then $\{\sigma_{\sigma,H} \mid g \in G\} \leq \mathbb{S}_X$ and $\{\sigma_{\sigma,H} \mid g \in G\}$ acts transitively on X
- * it is important to realize that the order of $\{\sigma_{g,H} \mid g \in G\}$ may be smaller than |G|, since the homomorphism $\Phi_H: g \to \sigma_{g,H}$ may not be injective; is it a homomorphism?
- * Prove that Φ_H is injective if and only if H is core-free in G, i.e..

$$\bigcap_{g \in G} gHg^{-1} = \langle 1_G \rangle$$

Induced action of a permutation group

Definition

Let $G \leq \mathbb{S}_X$, and $1 \leq k \leq |X|$. The induced action of G on $\mathcal{P}_k(X)$, the set of all k-element subsets of X, is the action

$$g\{x_1, x_2, \ldots, x_k\} = \{gx_1, gx_2, \ldots, gx_k\},\$$

defined for every $g \in G$ and every k-tuple $\{x_1, x_2, \dots, x_k\} \in \mathcal{P}_k(X)$.

- \star for every $g \in G$ the action of g on $\mathcal{P}_k(X)$ is a permutation of $\mathcal{P}_k(X)$
- * the set of all these permutations under the operation of composition is a subgroup of $\mathbb{S}_{\mathcal{P}_k(X)}$
- \star is this action transitive on $\mathcal{P}_k(X)$?

Construction of vertex-transitive graphs

Let $G \leq S_X$ acting transitively on X.

- * Let $\{u, v\}$ be an arbitrary pair from $\mathcal{P}_2(X)$. Then $(X, \{u, v\}^G)$ is a vertex-transitive graph; where $\{u, v\}^G$ is the orbit of $\{u, v\}$ under the induced action of G on $\mathcal{P}_2(X)$.
- * Let $\{u_i, v_i\}, i \in \mathcal{I}$, be a set of pairs from $\mathcal{P}_2(X)$. Then $(X, \bigcup_{i \in \mathcal{I}} \{u_i, v_i\}^G)$ is a vertex-transitive graph

Other Related Questions

- \star Is there an upper bound on the order of $Aut(\Gamma)$?
- * Is there an upper bound on the order of $Aut(\Gamma)$ that is a function of the order $|V(\Gamma)| = n$?
- * Is there an upper bound on the order of $Aut(\Gamma)$ for a connected 2-regular graph that is a function of the order $|V(\Gamma)| = n$ and that is better than the previous bound?
- \star Is there an upper bound on the order of $Aut(\Gamma)$ for a connected 3-regular graph that is a function of the order $|V(\Gamma)| = n$ and that is better than the previous bound?

Circulants

Definition

A graph $\Gamma = (V, E)$ of order n is called a **circulant** if $Aut(\Gamma)$ contains an automorphism φ such that φ forms a single cycle containing all of $V(\Gamma)$, i.e.,

$$\varphi: u_{i_1} \mapsto u_{i_2} \mapsto u_{i_3} \mapsto \cdots \mapsto u_{i_n} \mapsto u_{i_1},$$

for some ordering of the vertices of Γ .

- * A circulant is necessarily vertex-transitive.
- * Is a vertex-transitive graph necessarily a circulant?
- * Is a circulant necessarily Hamiltonian?

Circulants

- * Let \mathbb{Z}_n be a cyclic group of order n > 2
- * Let $S \subseteq \mathbb{Z}_n$ be a subset containing -k with every $k \in S$
- \star Consider the graph $\Gamma = C(\mathbb{Z}_n, S)$ with vertex set \mathbb{Z}_n and adjacency $k \sim i$ iff $k - s \in S$; we call such graph a **Cayley graph** of the group \mathbb{Z}_n with respect to the **connecting set** S
- * The graph $\Gamma = C(\mathbb{Z}_n, S)$ is necessarily a circulant, since $\varphi: \mathbb{Z}_n \to \mathbb{Z}_n, \ \varphi(k) = k+1 \pmod{n}$, is an automorphism of Γ that forms a single cycle.
- * The graph $\Gamma = C(\mathbb{Z}_n, S)$ is |S|-regular and vertex-transitive.

Theorem

A graph Γ of order n is a circulant if and only if $\Gamma = C(\mathbb{Z}_n, S)$ for some $S \subseteq \mathbb{Z}_n$.

A Generalization of Circulants

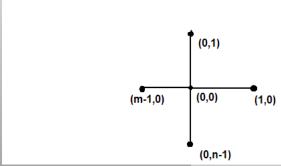
Definition

Given a group G, and a generating set $X = \{x_1, x_2, \dots, x_d\}$, $\langle X \rangle = G$, that is closed under taking inverses and does not contain 1_G , the vertices of the **Cayley graph** C(G,X) are the elements of the group G, and each vertex $g \in G$ is connected to the vertices gx_1, gx_2, \dots, gx_d .

Cayley Graph - An Example



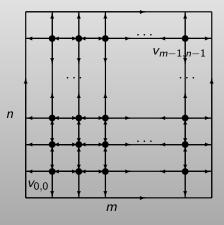
$$G = \mathbb{Z}_m \times \mathbb{Z}_n$$
, $X = \{(1,0), (0,1), (m-1,0), (n-1,0)\}$



obert Jajcay

Cayley Graph - An Example

$$G = \mathbb{Z}_m \times \mathbb{Z}_n, \qquad X = \{(1,0), (0,1), (m-1,0), (n-1,0)\}$$



Robert Jajcay Gra

Automorphism Groups of Cayley Graphs

- \star Let G be a finite group, and let $g \in G$.
- \star Let $\sigma_g: G \to G$ be the mapping defined via the formula $\sigma_g(h) = gh$
- \star The mapping σ_g is a bijection from G to G, for every $g\in G,$ i.e., $\sigma_g\in\mathbb{S}_G$
- * The mapping $\Phi: G \to \mathbb{S}_G$, $\Phi(g) = \sigma_g$, for every $g \in G$, is an injective homomorphism; hence $G_L = \{\sigma_g \mid g \in G\} \leq \mathbb{S}_G$ isomorphic to G

Theorem

- 1. C(G,X) is connected if and only if $\langle X \rangle = G$
- $2. \ G_L \leq C(G,X)$
- 3. C(G,X) is vertex-transitive for every finite group G and connecting set X
- 4. $Aut(G, X) = \{ \varphi \in Aut(G) \mid \varphi(X) = X \} = Stab_{Aut(G)}(\{X\}) \le Stab_{Aut(C(G,X))}(1_G)$

Automorphism Groups of Cayley Graphs

Theorem

- 1. If G is abelian, then $G_L < Aut(C(G,X))$ (i.e., $|Stab_{Aut(C(G,X))}(1_G)| \ge 2$)
- 2. $Aut(C(G,X)) = G_L$, for almost all Cayley graphs

Paley Graphs

- \star Let q be a prime power, $q \equiv 1 \pmod{4}$, \mathbb{F}_q be the finite field of order a
- * Let $G = (\mathbb{F}_q, +)$ and let S be the set of non-zero squares of \mathbb{F}_q
- * The Cayley graph (G, S) is called the **Paley graph** of order q
- * It is a $(q, \frac{1}{2}(q-1), \frac{1}{4}(q-5), \frac{1}{4}(q-1))$ -srg
- * It is often used as a 'random' graph

Highly Symmetric Graphs

- \star A graph Γ is **vertex-transitive** if $Aut(\Gamma)$ acts transitively on the vertices of Γ , i.e., for all pairs of vertices $u, v \in V(\Gamma)$ there exists an automorphism $\varphi \in Aut(\Gamma)$ such that $\varphi(u) = v$
- \star A graph Γ is **edge-transitive** if $Aut(\Gamma)$ acts transitively on the edges of Γ , i.e., for all pairs of edges $\{u, v\}, \{s, t\} \in E(\Gamma)$ there exists an automorphism $\varphi \in Aut(\Gamma)$ such that $\varphi(\{u,v\}) = \{s,t\}$
- \star A graph Γ is **arc-transitive** (or **symmetric**) if $Aut(\Gamma)$ acts transitively on the arcs of Γ , i.e., for any two pairs of adjacent vertices $(u, v), (s, t) \in D(\Gamma)$ there exists an automorphism $\varphi \in Aut(\Gamma)$ such that $\varphi(u) = s, \varphi(v) = t$

Highly Symmetric Graphs

Recall that if G acts transitively on a set X, then |X| divides |G|. Hence.

- \star if Γ is vertex-transitive, $|Aut(\Gamma)| \geq |V(\Gamma)|$
- \star if Γ is edge-transitive, $|Aut(\Gamma)| \geq |E(\Gamma)|$
- \star if Γ is arc-transitive, $|Aut(\Gamma)| \geq 2|E(\Gamma)|$
- $\star K_n$ is vertex-, edge- and arc-transitive (and so is K_n)
- $\star K_{m,n}$ is edge-transitive, but not vertex-transitive or arc-transitive if $m \neq n$
- $\star K_{m,m}$ is vertex-, edge- and arc-transitive
- * arc-transitivity implies vertex- and edge-transitivity
- * the order of a vertex-stabilizer of an arc-transitive cubic graph is of order at most 48

s-arc-transitive graphs

Definition

- 1. Let Γ be a graph. An s-arc of Γ is a set of vertices u_0, u_1, \ldots, u_s such that $u_i \sim u_{i+1}$, for all $0 \leq i < s$, and $u_i \neq u_{i+2}$, for all $0 \leq i < s-1$.
- 2. A graph Γ is s-arc-transitive if $Aut(\Gamma)$ acts transitively on the s-arcs of Γ , i.e., for any two s-arcs u_0, u_1, \ldots, u_s , v_0, v_1, \ldots, v_s there exists an automorphism $\varphi \in Aut(\Gamma)$ such that $\varphi(u_i) = v_i$, for all $0 \le i \le s$.
- \star C_n is s-arc-transitive for every $0 \le s$
- $\star~ \mathcal{K}_n$ (and $ilde{\mathcal{K}}_n$) is s-arc-transitive for every $0 \leq s \leq n$
- \star if Γ is s-arc-transitive for s>7, then Γ is C_n or \tilde{K}_n or \tilde{K}_n

