Graph Theory
Lecture #12
Highly symmetric graphs IV
Spectral Graph Theory

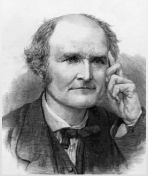
Robert Jajcay

December 12, 2024

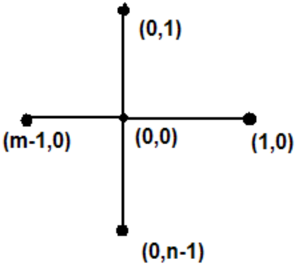
Definition

Given a group G, and a generating set $X = \{x_1, x_2, \ldots, x_d\}$, $\langle X \rangle = G$, that is closed under taking inverses and does not contain 1_G, the vertices of the **Cayley graph** $C(G, X)$ are the elements of the group G, and each vertex $g \in G$ is connected to the vertices $gx_1, gx_2, \ldots, gx_d$.

Cayley Graph - An Example

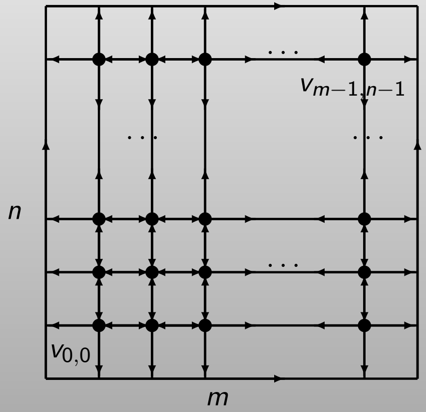


$G = \mathbb{Z}_m \times \mathbb{Z}_n$, $X = \{(1,0),(0,1),(m-1,0),(n-1,0)\}$



Cayley Graph - An Example

$$G = \mathbb{Z}_m \times \mathbb{Z}_n, \qquad X = \{(1,0),(0,1),(m-1,0),(n-1,0)\}$$



Automorphism Groups of Cayley Graphs

- ⋆ Let G be a finite group, and let $g \in G$.
- ⋆ Let $\sigma_g : G \to G$ be the mapping defined via the formula $\sigma_g(h) = gh$
- ⋆ The mapping σ_g is a bijection from G to G, for every $g \in G$, i.e., $\sigma_g \in \mathbb{S}_G$
- ⋆ The mapping $\Phi : G \to \mathbb{S}_G$, $\Phi(g) = \sigma_g$, for every $g \in G$, is an injective homomorphism; hence $G_L = \{\sigma_g \mid g \in G\} \leq \mathbb{S}_G$ is isomorphic to G

Theorem

1. *$C(G, X)$ is connected if and only if $\langle X \rangle = G$*
2. $G_L \leq C(G, X)$
3. *$C(G, X)$ is vertex-transitive for every finite group G and connecting set X*
4. $Aut(G, X) = \{\varphi \in Aut(G) \mid \varphi(X) = X\} = Stab_{Aut(G)}(\{X\}) \leq Stab_{Aut(C(G,X))}(1_G)$

Automorphism Groups of Cayley Graphs

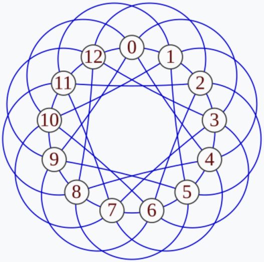
Theorem

1. *If G is abelian, then $G_L < Aut(C(G,X))$ (i.e., $|Stab_{Aut(C(G,X))}(1_G)| \geq 2$)*
2. *$Aut(C(G,X)) = G_L$, for almost all Cayley graphs*

Paley Graphs

- Let q be a prime power, $q \equiv 1 \pmod 4$, $\mathbb{F}_q$ be the finite field of order q
- Let $G = (\mathbb{F}_q, +)$ and let S be the set of non-zero squares of $\mathbb{F}_q$
- The Cayley graph (G, S) is called the **Paley graph** of order q
- It is a $(q, \frac{1}{2}(q-1), \frac{1}{4}(q-5), \frac{1}{4}(q-1))$-srg
- It is often used as a 'random' graph

Paley graph



The Paley graph of order 13

Highly Symmetric Graphs

- A graph Γ is **vertex-transitive** if $Aut(\Gamma)$ acts transitively on the vertices of Γ, i.e., for all pairs of vertices $u, v \in V(\Gamma)$ there exists an automorphism $\varphi \in Aut(\Gamma)$ such that $\varphi(u) = v$
- A graph Γ is **edge-transitive** if $Aut(\Gamma)$ acts transitively on the edges of Γ, i.e., for all pairs of edges $\{u, v\}, \{s, t\} \in E(\Gamma)$ there exists an automorphism $\varphi \in Aut(\Gamma)$ such that $\varphi(\{u, v\}) = \{s, t\}$
- A graph Γ is **arc-transitive** (or **symmetric**) if $Aut(\Gamma)$ acts transitively on the arcs of Γ, i.e., for any two pairs of *adjacent vertices* $(u, v), (s, t) \in D(\Gamma)$ there exists an automorphism $\varphi \in Aut(\Gamma)$ such that $\varphi(u) = s, \varphi(v) = t$

Highly Symmetric Graphs

Recall that if G acts transitively on a set X, then $|X|$ divides $|G|$. Hence,

- if Γ is vertex-transitive, $|Aut(\Gamma)| \geq |V(\Gamma)|$
- if Γ is edge-transitive, $|Aut(\Gamma)| \geq |E(\Gamma)|$
- if Γ is arc-transitive, $|Aut(\Gamma)| \geq 2|E(\Gamma)|$

- K_n is vertex-, edge- and arc-transitive (and so is $\tilde{K}_n$)
- $K_{m,n}$ is edge-transitive, but not vertex-transitive or arc-transitive if $m \neq n$
- $K_{m,m}$ is vertex-, edge- and arc-transitive
- arc-transitivity implies vertex- and edge-transitivity
- the order of a vertex-stabilizer of an arc-transitive cubic graph is of order at most 48

s-arc-transitive graphs

Definition

1. Let Γ be a graph. An s-**arc** of Γ is a set of vertices $u_0, u_1, \ldots, u_s$ such that $u_i \sim u_{i+1}$, for all $0 \leq i < s$, and $u_i \neq u_{i+2}$, for all $0 \leq i < s-1$.
2. A graph Γ is s-**arc-transitive** if $Aut(\Gamma)$ acts transitively on the s-arcs of Γ, i.e., for any two s-arcs $u_0, u_1, \ldots, u_s$, $v_0, v_1, \ldots, v_s$ there exists an automorphism $\varphi \in Aut(\Gamma)$ such that $\varphi(u_i) = v_i$, for all $0 \leq i \leq s$.

- $\star$ C_n is s-arc-transitive for every $0 \leq s$
- $\star$ K_n (and $\tilde{K}_n$) is s-arc-transitive for every $0 \leq s \leq n$
- $\star$ if Γ is s-arc-transitive for $s > 7$, then Γ is C_n or K_n or $\tilde{K}_n$

Spectral Graph Theory

The Adjacency Matrix of a Graph

Definition

Let $\Gamma = (V, E)$ be a graph; $V = \{v_1, v_2, \ldots, v_n\}$. The $0, 1$ $n \times n$ matrix $\mathbf{A}$ defined via the rule

$$a_{ij} = 1 \text{ if and only if } v_i \sim v_j, \text{ otherwise } a_{ij} = 0$$

is called the adjacency matrix of Γ.

Note that the adjacency matrix depends on the ordering of the vertices of Γ:
If $\varphi \in Sym_n$, the adjacency matrix of Γ with respect to the ordering of vertices $v_{\varphi(1)}, v_{\varphi(2)}, \ldots, v_{\varphi(n)}$ is the matrix obtained from Γ by simultaneous reordering of the rows and columns of $\mathbf{A}$ determined by φ.

The Adjacency Matrices of a Graph

Let $\mathbf{P}$ be the **permutation matrix** of φ, i.e.,

$$p_{ij} = 1 \text{ if and only if } \varphi(i) = j, \text{ otherwise } a_{ij} = 0$$

Then, the adjacency matrix of Γ with respect to the ordering of vertices $v_{\varphi(1)}, v_{\varphi(2)}, \ldots, v_{\varphi(n)}$ is the matrix

$$\mathbf{PAP}^T$$

Note that $\mathbf{P}^T$ is the permutation matrix of φ^{-1}, and hence $\mathbf{P}^T = \mathbf{P}^{-1}$.
I.e., the adjacency matrix of Γ with respect to the ordering of vertices $v_{\varphi(1)}, v_{\varphi(2)}, \ldots, v_{\varphi(n)}$ is the matrix

$$\mathbf{PAP}^{-1}$$

and therefore, **all the adjacency matrices of Γ are mutual conjugates**.

The Eigenvalues of Γ

Since conjugate matrices $\mathbf{A}$ and $\mathbf{PAP}^{-1}$ have the same characteristic polynomial:

$$char_{\mathbf{PAP}^{-1}}(x) = \det(\mathbf{PAP}^{-1} - x\mathbb{I}_n) = \det(\mathbf{PAP}^{-1} - x\mathbf{P}\mathbb{I}_n\mathbf{P}^{-1})$$

$$= \det(\mathbf{P}(\mathbf{A} - x\mathbb{I}_n)\mathbf{P}^{-1}) = \det(\mathbf{P})\det(\mathbf{A} - x\mathbb{I}_n)\det(\mathbf{P}^{-1}) = char_{\mathbf{A}}(x),$$

All adjacency matrices of Γ (regardless of the ordering of the vertices of Γ) have the same eigenvalues.

Since $\mathbf{A}$ is real and symmetric, its eigenvalues are reals and their multiplicities are the dimensions of their corresponding eigenspaces.

The Spectrum of Eigenvalues of Γ

Definition

The **spectrum** of a graph Γ is the set of eigenvalues of the adjacency matrix of Γ together with their multiplicities.
If the distinct eigenvalues of the adjacency matrix of Γ are $\lambda_0 > \lambda_1 > \ldots > \lambda_{s-1}$, of respective multiplicities $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{s-1})$, we denote

$$Spec\Gamma = \lambda_0^{m(\lambda_0)} \lambda_1^{m(\lambda_1)} \ldots \lambda_{s-1}^{m(\lambda_{s-1})}$$

Example: The adjacency matrix of K_4 is

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$SpecK_4 = 3^1(-1)^3$$

The Characteristic Polynomial of Γ

Lemma

The coefficients of the characteristic polynomial of a graph Γ:

$$char_{\Gamma}(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_{n-1} x + c_n$$

satisfy

1. $c_1 = 0$
2. $-c_2$ *is the number of edges of* Γ
3. $-c_3$ *is twice the number of triangles in* Γ

Proof:

1. Recall that the coefficients of the characteristic polynomial of a matrix can be expressed in terms of **principal minors**.
2. A principal minor of a matrix $\mathbf{A}$ is the determinant of a submatrix obtained from $\mathbf{A}$ by erasing a set of corresponding rows and columns

The Characteristic Polynomial of Γ

1. $c_1 = 0$
2. $-c_2$ is the number of edges of Γ
3. $-c_3$ is twice the number of triangles in Γ

Proof:

1. For each i, $(-1)^i c_i$ is equal to the sum of principal minors with i rows and columns
2. Thus, $-c_1$ is the sum of the elements on the main diagonal, which is also the **trace** of the adjacency matrix, and is equal to 0
3. The coefficient c_2 is equal to the sum of determinants of matrices that are intersections of i-th and j-th row and i-th and j-th column. They are either equal to an all-zero matrix or are of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with the second kind occurring if and only if $v_i \sim v_j$ (and having the determinant -1). Thus, $-c_2$ is the number of edges of Γ

The Characteristic Polynomial of Γ

1. $c_1 = 0$
2. $-c_2$ is the number of edges of Γ
3. $-c_3$ is twice the number of triangles in Γ

Proof:

1. $-c_3$ is the sum of principal minors with 3 rows and columns
2. the non-trivial ones come in three kinds

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

3. the non-zero determinant is equal to 2 and represents a triangle

$\square$

More Algebra

Suppose $\mathbf{A}$ is the adjacency matrix of a graph Γ. Then the set of polynomials in $\mathbf{A}$, with complex coefficients, forms an algebra under the usual matrix operations. This algebra has finite dimension as a complex vector space. Indeed, the Cayley–Hamilton theorem asserts that $\mathbf{A}$ satisfies its own characteristic equation, so the dimension is at most n, the number of vertices in Γ.

Definition 2.4 The *adjacency algebra* of a graph Γ is the algebra of polynomials in the adjacency matrix $\mathbf{A} = \mathbf{A}(\Gamma)$. We shall denote the adjacency algebra of Γ by $\mathcal{A}(\Gamma)$.

More Algebra

1. The powers of the adjacency matrix span the algebra $\mathcal{A}(\Gamma)$
2. a **walk** in Γ of length ℓ joining vertices v_i and v_j is a sequence $v_i = u_0, u_1, \ldots, u_\ell = v_j$ in which $u_i \sim u_{i+1}$, for all $0 \leq i < \ell$

Lemma

If $\mathbf{A}$ *is the adjacency matrix of* Γ, *the* i, j *entry in* $\mathbf{A}^\ell$ *is the number of walks of length* ℓ *joining* v_i *and* v_j.

1. Can one determine the diameter of Γ from the powers of $\mathbf{A}(\Gamma)$?

Lemma

Let Γ *be a connected graph with adjacency algebra* $\mathcal{A}(\Gamma)$ *and diameter* d. *Then the dimension of* $\mathcal{A}(\Gamma)$ *is at least* $d + 1$.

More Algebra

If $\mathbf{A}(\Gamma)$ has s distinct eigenvalues $\Longrightarrow$ since $\mathbf{A}(\Gamma)$ is real symmetric, the minimal polynomial of $\mathcal{A}(\Gamma)$ is of degree $s \Longrightarrow$ the dimension of $\mathcal{A}(\Gamma)$ is $s \Longrightarrow$

Corollary

A connected Γ of diameter d has at least $d+1$ distinct eigenvalues.

Circulants Revisited

1. a **circulant matrix** is a square matrix in which every next row is obtained by a cyclic shift of the previous row; i.e., a circulant matrix is determined by its first row
2. let $\mathbf{W}$ be the $n \times n$ circulant matrix whose first row is $(0, 1, 0, 0, \ldots, 0)$
3. the characteristic polynomial of $\mathbf{W}$ is $x^n - 1$
4. the eigenvalues of $\mathbf{W}$ are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = \cos(\frac{2\pi}{n}) + i \sin(\frac{2\pi}{n})$
5. Γ is a circulant if and only if its vertices can be ordered in such a way that $\mathbf{A}(\Gamma)$ is a circulant matrix, i.e., Γ is a circulant if and only if there exists a permutation matrix $\mathbf{P}$ such that $\mathbf{P}\mathbf{A}(\Gamma)\mathbf{P}^T$ is a circulant matrix

Circulants Revisited

1. if $\mathbf{S}$ is the circulant matrix with the first row $(s_1, s_2, \ldots, s_n)$, then
$$\mathbf{S} = \sum_{j=1}^{n} s_j \mathbf{W}^{j-1}$$
2. if $\mathbf{S}$ is the circulant matrix with the first row $(s_1, s_2, \ldots, s_n)$, the eigenvalues of $\mathbf{S}$ are
$$\lambda_r = \sum_{j=1}^{n} s_j \omega^{(j-1)r}, 0 \leq r < n$$
3. Determine the spectrum of K_n
4. Determine the spectrum of C_n
5. Determine the spectrum of the **cock-tail party graph** K_{2s} with 1-factor removed

Automorphisms Revisited

1. a permutation π of $V(\Gamma)$ can be represented by the $0-1$ **permutation matrix** $\mathbf{P}_\pi$: $p_{ij}=1$ iff $\pi(i)=j$.

Lemma

π is an automorphism of Γ if and only

$$\mathbf{P}_\pi\mathbf{A}(\Gamma)=\mathbf{A}(\Gamma)\mathbf{P}_\pi$$

i.e., the automorphism group of Γ is isomorphic to the group of permutation matrices which commute with $\mathbf{A}(\Gamma)$
i.e., the automorphism group of Γ is isomorphic to the stabilizer of $\mathbf{A}(\Gamma)$ under the conjugation action of the group of permutation matrices (which is isomorphic to $\mathbb{S}_n$).

Automorphisms Revisited

1. let $\vec{x}$ be an eigenvector of $\mathbf{A}(\Gamma)$ corresponding to the eigenvalue λ, then
2. $$\mathbf{A}(\Gamma)\mathbf{P}_\pi\vec{x} = \mathbf{P}_\pi\mathbf{A}(\Gamma)\vec{x} = \mathbf{P}_\pi\lambda\vec{x} = \lambda\mathbf{P}_\pi\vec{x}$$
3. and $\mathbf{P}_\pi\vec{x}$ is also an eigenvector of $\mathbf{A}(\Gamma)$ corresponding to λ, thus
4. if $\vec{x}$ and $\mathbf{P}_\pi\vec{x}$ are linearly independent, λ is not a simple eigenvalue

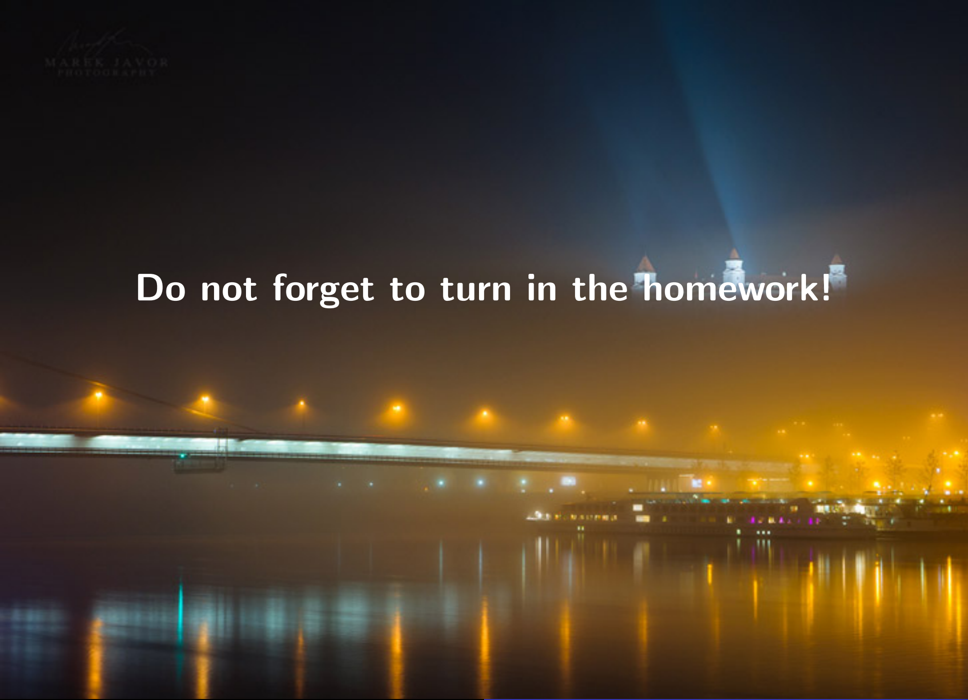
Automorphisms Revisited

Lemma 15.3 *Let λ be an simple eigenvalue of Γ, and let $\mathbf{x}$ be a corresponding eigenvector, with real components. If the permutation matrix $\mathbf{P}$ represents an automorphism of Γ then $\mathbf{Px} = \pm\mathbf{x}$.*

Proof If λ has multiplicity one, $\mathbf{x}$ and $\mathbf{Px}$ are linearly dependent; that is $\mathbf{Px} = \mu\mathbf{x}$ for some complex number μ. Since $\mathbf{x}$ and $\mathbf{P}$ are real, μ is real; and, since $\mathbf{P}^s = \mathbf{I}$ for some natural number $s \geq 1$, it follows that μ is an sth root of unity. Consequently $\mu = \pm 1$ and the lemma is proved. □

Theorem 15.4 (Mowshowitz 1969, Petersdorf and Sachs 1969) *If all the eigenvalues of the graph Γ are simple, every automorphism of Γ (apart from the identity) has order* 2.

Proof Suppose that every eigenvalue of Γ has multiplicity one. Then, for any permutation matrix $\mathbf{P}$ representing an automorphism of Γ, and any eigenvector $\mathbf{x}$, we have $\mathbf{P}^2\mathbf{x} = \mathbf{x}$. The space spanned by the eigenvectors is the whole space of column vectors, and so $\mathbf{P}^2 = \mathbf{I}$. □



Do not forget to turn in the homework!