Graph Theory
Lecture #13
Spectral Graph Theory

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Záverečné skúšky

- **Termíny skúšok sú: 7.1., 9.1., 13.1., 15.1. a 17.1, 2025**
- Všetky písomky sa konajú v **M-X**, okrem 9.1., kedy je písomka v M-XI
- **Dohodli sme sa, že Vy prídete 13.1.**
- **termíny sú vždy o 9:00**
- **písomka od 9:00 do 11:00**
- **ústna časť od 12:00 do 16:30; alebo skôr**
- **treba mi poslať e-mail**

The Adjacency Matrix of a Graph

Definition

Let $\Gamma = (V, E)$ be a graph; $V = \{v_1, v_2, \ldots, v_n\}$. The $0, 1$ $n \times n$ matrix $\mathbf{A}$ defined via the rule

$$a_{ij} = 1 \text{ if and only if } v_i \sim v_j, \text{ otherwise } a_{ij} = 0$$

is called the adjacency matrix of Γ.

Let $\mathbf{P}$ be the **permutation matrix** of φ, i.e.,

$$p_{ij} = 1 \text{ if and only if } \varphi(i) = j, \text{ otherwise } a_{ij} = 0$$

Then, the adjacency matrix of Γ with respect to the ordering of vertices $v_{\varphi(1)}, v_{\varphi(2)}, \ldots, v_{\varphi(n)}$ is the matrix

$$\mathbf{PAP}^{-1}$$

The Spectrum of Eigenvalues of Γ

Definition

The **spectrum** of a graph Γ is the set of eigenvalues of the adjacency matrix of Γ together with their multiplicities.
If the distinct eigenvalues of the adjacency matrix of Γ are $\lambda_0 > \lambda_1 > \ldots > \lambda_{s-1}$, of respective multiplicities $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{s-1})$, we denote

$$Spec\Gamma = \lambda_0^{m(\lambda_0)} \lambda_1^{m(\lambda_1)} \ldots \lambda_{s-1}^{m(\lambda_{s-1})}$$

The Characteristic Polynomial of Γ

Lemma

The coefficients of the characteristic polynomial of a graph Γ:

$$char_\Gamma(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_{n-1} x + c_n$$

satisfy

1. $c_1 = 0$
2. $-c_2$ *is the number of edges of* Γ
3. $-c_3$ *is twice the number of triangles in* Γ

More Algebra

Suppose $\mathbf{A}$ is the adjacency matrix of a graph Γ. Then the set of polynomials in $\mathbf{A}$, with complex coefficients, forms an algebra under the usual matrix operations. This algebra has finite dimension as a complex vector space. Indeed, the Cayley–Hamilton theorem asserts that $\mathbf{A}$ satisfies its own characteristic equation, so the dimension is at most n, the number of vertices in Γ.

Definition 2.4 The *adjacency algebra* of a graph Γ is the algebra of polynomials in the adjacency matrix $\mathbf{A} = \mathbf{A}(\Gamma)$. We shall denote the adjacency algebra of Γ by $\mathcal{A}(\Gamma)$.

More Algebra

1. The powers of the adjacency matrix span the algebra $\mathcal{A}(\Gamma)$
2. a **walk** in Γ of length ℓ joining vertices v_i and v_j is a sequence $v_i = u_0, u_1, \ldots, u_\ell = v_j$ in which $u_i \sim u_{i+1}$, for all $0 \leq i < \ell$

Lemma

If $\mathbf{A}$ is the adjacency matrix of Γ, the i, j entry in $\mathbf{A}^\ell$ is the number of walks of length ℓ joining v_i and v_j.

Lemma

Let Γ be a connected graph with adjacency algebra $\mathcal{A}(\Gamma)$ and diameter d. Then the dimension of $\mathcal{A}(\Gamma)$ is at least $d + 1$.

More Algebra

If $\mathbf{A}(\Gamma)$ has s distinct eigenvalues $\Longrightarrow$ since $\mathbf{A}(\Gamma)$ is real symmetric, the minimal polynomial of $\mathcal{A}(\Gamma)$ is of degree $s \Longrightarrow$ the dimension of $\mathcal{A}(\Gamma)$ is $s \Longrightarrow$

Corollary

A connected Γ of diameter d has at least $d+1$ distinct eigenvalues.

Circulants Revisited

1. a **circulant matrix** is a square matrix in which every next row is obtained by a cyclic shift of the previous row; i.e., a circulant matrix is determined by its first row
2. let $\mathbf{W}$ be the $n \times n$ circulant matrix whose first row is $(0, 1, 0, 0, \ldots, 0)$
3. the characteristic polynomial of $\mathbf{W}$ is $x^n - 1$
4. the eigenvalues of $\mathbf{W}$ are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = \cos(\frac{2\pi}{n}) + i \sin(\frac{2\pi}{n})$
5. Γ is a circulant if and only if its vertices can be ordered in such a way that $\mathbf{A}(\Gamma)$ is a circulant matrix, i.e., Γ is a circulant if and only if there exists a permutation matrix $\mathbf{P}$ such that $\mathbf{PA}(\Gamma)\mathbf{P}^T$ is a circulant matrix

Circulants Revisited

1. if $\mathbf{S}$ is the circulant matrix with the first row $(s_1, s_2, \ldots, s_n)$, then
$$\mathbf{S} = \sum_{j=1}^{n} s_j \mathbf{W}^{j-1}$$
2. if $\mathbf{S}$ is the circulant matrix with the first row $(s_1, s_2, \ldots, s_n)$, the eigenvalues of $\mathbf{S}$ are
$$\lambda_r = \sum_{j=1}^{n} s_j \omega^{(j-1)r}, 0 \leq r < n$$
3. Determine the spectrum of K_n
4. Determine the spectrum of C_n
5. Determine the spectrum of the **cock-tail party graph** K_{2s} with 1-factor removed

Automorphisms Revisited

1. a permutation π of $V(\Gamma)$ can be represented by the $0-1$ **permutation matrix $\mathbf{P}_\pi$**: $p_{ij} = 1$ iff $\pi(i) = j$.

Lemma

π is an automorphism of Γ if and only

$$\mathbf{P}_\pi \mathbf{A}(\Gamma) = \mathbf{A}(\Gamma)\mathbf{P}_\pi$$

i.e., the automorphism group of Γ is isomorphic to the group of permutation matrices which commute with $\mathbf{A}(\Gamma)$
i.e., the automorphism group of Γ is isomorphic to the stabilizer of $\mathbf{A}(\Gamma)$ under the conjugation action of the group of permutation matrices (which is isomorphic to $\mathbb{S}_n$).

Automorphisms Revisited

1. let $\vec{x}$ be an eigenvector of $\mathbf{A}(\Gamma)$ corresponding to the eigenvalue λ, then
2. $$\mathbf{A}(\Gamma)\mathbf{P}_\pi\vec{x} = \mathbf{P}_\pi\mathbf{A}(\Gamma)\vec{x} = \mathbf{P}_\pi\lambda\vec{x} = \lambda\mathbf{P}_\pi\vec{x}$$
3. and $\mathbf{P}_\pi\vec{x}$ is also an eigenvector of $\mathbf{A}(\Gamma)$ corresponding to λ, thus
4. if $\vec{x}$ and $\mathbf{P}_\pi\vec{x}$ are linearly independent, λ is not a simple eigenvalue

Automorphisms Revisited

Lemma 15.3 *Let λ be an simple eigenvalue of Γ, and let $\mathbf{x}$ be a corresponding eigenvector, with real components. If the permutation matrix $\mathbf{P}$ represents an automorphism of Γ then $\mathbf{Px} = \pm\mathbf{x}$.*

Proof If λ has multiplicity one, $\mathbf{x}$ and $\mathbf{Px}$ are linearly dependent; that is $\mathbf{Px} = \mu\mathbf{x}$ for some complex number μ. Since $\mathbf{x}$ and $\mathbf{P}$ are real, μ is real; and, since $\mathbf{P}^s = \mathbf{I}$ for some natural number $s \geq 1$, it follows that μ is an sth root of unity. Consequently $\mu = \pm 1$ and the lemma is proved. □

Theorem 15.4 (Mowshowitz 1969, Petersdorf and Sachs 1969) *If all the eigenvalues of the graph Γ are simple, every automorphism of Γ (apart from the identity) has order* 2.

Proof Suppose that every eigenvalue of Γ has multiplicity one. Then, for any permutation matrix $\mathbf{P}$ representing an automorphism of Γ, and any eigenvector $\mathbf{x}$, we have $\mathbf{P}^2\mathbf{x} = \mathbf{x}$. The space spanned by the eigenvectors is the whole space of column vectors, and so $\mathbf{P}^2 = \mathbf{I}$. □

Strongly Regular Graphs Revisited

A *strongly regular graph* $srg(v, k, \lambda, \mu)$ is a graph with v vertices that is regular of degree k and that has the following properties:

(1) For any two adjacent vertices x, y, there are exactly λ vertices adjacent to x and to y.

(2) For any two nonadjacent vertices x, y, there are exactly μ vertices adjacent to x and to y.

A trivial example is a pentagon, an $srg(5, 2, 0, 1)$. Perhaps the most famous example is the graph of Fig. 1.4, the Petersen graph, an $srg(10, 3, 0, 1)$.

The complement of a strongly regular graph with parameters (v, k, λ, μ) has parameters

$$srg(v, v-k-1, v-2k+\mu-2, v-2k+\lambda) \tag{21.2}$$

and since the parameters are nonnegative, we find a simple condition on the parameters, namely

$$v - 2k + \mu - 2 \geq 0. \tag{21.3}$$

Strongly Regular Graphs Revisited

Another relation between the parameters is easily found as follows. Consider any vertex x and partition the other vertices into the set $\Gamma(x)$ of vertices joined to x and the set $\Delta(x)$ of vertices not joined to x. By the definition of strongly regular graphs, $\Gamma(x)$ consists of k vertices, each of which is joined to λ vertices of $\Gamma(x)$. Each vertex in $\Delta(x)$ is joined to μ vertices in $\Gamma(x)$. Counting edges with one end in $\Gamma(x)$ and one end in $\Delta(x)$ in two ways, we find

$$k(k-\lambda-1)=\mu(v-k-1). \qquad (21.4)$$

(21.3) and (21.4) are **necessary conditions that are not sufficient** for the existence of an srg-(v, k, λ, μ)

One more example:

Example 21.4. The Clebsch graph has as vertices all subsets of even cardinality of the set $\{1,\ldots,5\}$; two vertices are joined if and only if their symmetric difference has cardinality 4. This is an $srg(16,5,0,2)$. One can also describe the vertices as the words of even weight in $\mathbb{F}_2^5$, with an edge if the distance is 4. For any vertex x, the induced subgraph on $\Delta(x)$ is the Petersen graph. See Fig. 21.1 below.

The Clebsch Graph

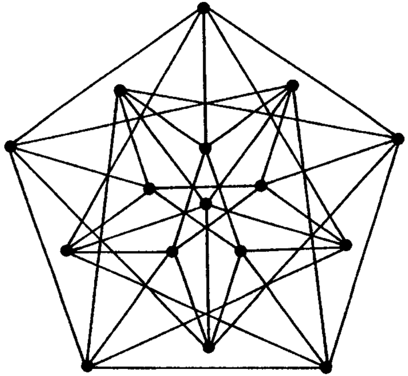

Figure 21.1

Strongly Regular Graphs Revisited

Let us define the *adjacency matrix* A of a graph G with v vertices $1, \ldots, v$ to be the $v \times v$ (0,1)-matrix with $a_{ij} = a_{ji} = 1$ if and only if the vertices i and j are joined. Clearly A is symmetric with zeros on the diagonal. The statement that G is an $srg(v, k, \lambda, \mu)$ is equivalent to

$$(21.5) \qquad AJ = kJ, \qquad A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J.$$

We define the *Bose–Mesner algebra* $\mathfrak{A}$ of an *srg* G to be the 3-dimensional algebra $\mathfrak{A}$ of linear combinations of I, J, and A. That this is indeed an algebra is a consequence of (21.5). This algebra consists of symmetric commuting matrices and therefore there is an orthogonal matrix that simultaneously diagonalizes them. This can also be seen in an elementary way from (21.5). In fact, we shall see in the next theorem that A has three distinct eigenspaces in $\mathbb{R}^v$ and each of them is an eigenspace for any element of $\mathfrak{A}$.

Strongly Regular Graphs Revisited

Theorem 21.1. *If there is an $srg(v, k, \lambda, \mu)$, then the numbers*

$$f := \frac{1}{2}\left\{v - 1 + \frac{(v-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\right\}$$

and

$$g := \frac{1}{2}\left\{v - 1 - \frac{(v-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\right\}$$

are nonnegative integers.

PROOF: Let A be the adjacency matrix of the graph. By (21.5) the all-one vector $\mathbf{j} := (1, 1, \ldots, 1)^\top$ is an eigenvector of A, with eigenvalue k, and of course it is also an eigenvector of I and of J. Application of (21.5) yields a second proof of (21.4). The multiplicity of this eigenvalue is one because the graph is connected. Any other eigenvector, say with eigenvalue x, is orthogonal to $\mathbf{j}$ and therefore we find from (21.5),

$$x^2 + (\mu - \lambda)x + (\mu - k) = 0.$$

Strongly Regular Graphs Revisited

This equation has two solutions

$$r, s = \frac{1}{2}\left\{\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right\} \tag{21.6}$$

Let f and g be the multiplicities of r and s as eigenvalues of A. Then we have

$$1 + f + g = v \quad \text{and} \quad tr(A) = k + fr + gs = 0.$$

If we solve these two linear equations, we find the assertion of the theorem. □

Strongly Regular Graphs Revisited

From (21.6) we can draw a further (surprising) conclusion. If $f \neq g$, then the square root in the denominator of the expressions for f and for g must be an integer, i.e. $(\mu - \lambda)^2 + 4(k - \mu)$ is a perfect square. It then follows from (21.6) that the eigenvalues r and s are *integers*!

The other case, i.e. when $f = g$, is usually called the *half-case*. We then have an $srg(4\mu + 1, 2\mu, \mu - 1, \mu)$. The Paley graphs are examples of the half-case. In his paper on conference telephony, mentioned in Chapter 18, Belevitch observed that a necessary condition for the existence of a conference matrix of order n is that $n - 1$ is the sum of two squares. See Problem 19M. We note that the parameters $v = 21$, $k = 10$, $\lambda = 4$, $\mu = 5$ satisfy all the necessary conditions for the existence of a strongly regular graph that we stated above, but the graph does not exist because, using (18.5), it would imply the existence of a conference matrix of order 22 and since 21 is not the sum of two squares, this is impossible.

Strongly Regular Graphs Revisited

The condition of Theorem 21.1 is known as the *integrality condition*. We shall call a parameter set (v, k, λ, μ) that satisfies these conditions and the earlier necessary conditions, a *feasible* set.

Problem 21B. Show that if an $srg(k^2 + 1, k, 0, 1)$ exists, then $k = 1, 2, 3, 7$ or 57. (See the notes to Chapter 4.)

Merry Christmas and a Happy New Year