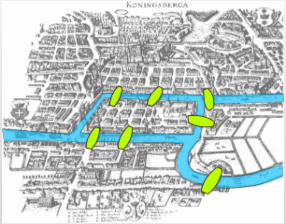
Graph Theory
2nd lecture
Eulerian and Hamiltonian Circuits

Robert Jajcay

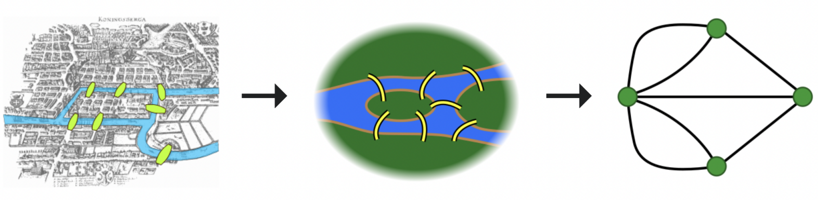
3.10.2024

Eulerian Circuit

Leonard Euler
(1707 - 1783)



Königsberg, 18th century.



Eulerian Circuit

Problem: Is it possible to walk through all the bridges, visiting each bridge exactly once, and ending at the same point where we started?

Definition

A closed path through a graph using every edge exactly once is called an **Eulerian circuit**.

Eulerian Circuit and Euler's Theorem

Definition

A closed path through a graph using every edge exactly once is called an **Eulerian circuit**.

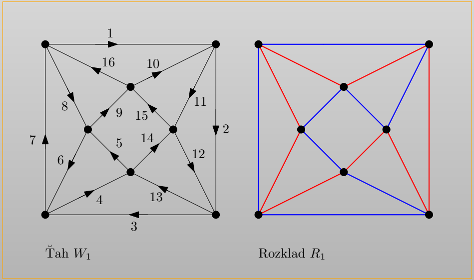
Theorem

A finite graph with no isolated vertices is Eulerian if and only if it is connected and every vertex has even degree.

Proof?

But a theorem like that does not have to mean the end of the story ...

Conjecture: (L. Sabidusi) Let G be an Eulerian graph of minimum degree at least 4 and let W be a closed Eulerian circuit in G. Then there exists a partition of the edges of G into cycles such that none of the cycles contains two consecutive edges that are also consecutive in W.



Sabidusi's Compatibility Conjecture

Sabidusi's Compatibility Conjecture

ELSEVIER

Discrete Mathematics 233 (2001) 247–256

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Stable dominating circuits in snarks

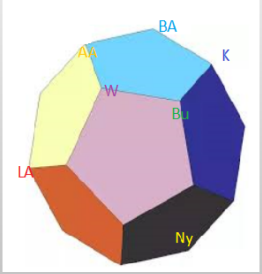
Martin Kochol

MÚ SAV, Štefánikova 49, 814 73 Bratislava 1, Slovakia

Abstract

Snarks are cyclically 4-edge-connected cubic graphs with girth at least 5 and with no 3-edge-coloring. We construct snarks with a (dominating) circuit C so that no other circuit C' satisfies $V(C) \subseteq V(C')$. These graphs are of interest because two known conjectures about graphs can be reduced on them. The first one is Sabidusi's Compatibility Conjecture which suggests that given an eulerian trail T in an eulerian graph G without 2-valent vertices, there exists a decomposition of G into circuits such that consecutive edges in T belong to different circuits. The second conjecture is the Fixed-Circuit Cycle Double-Cover Conjecture suggesting that every bridgeless graph has a cycle double cover which includes a fixed circuit.

Around the World

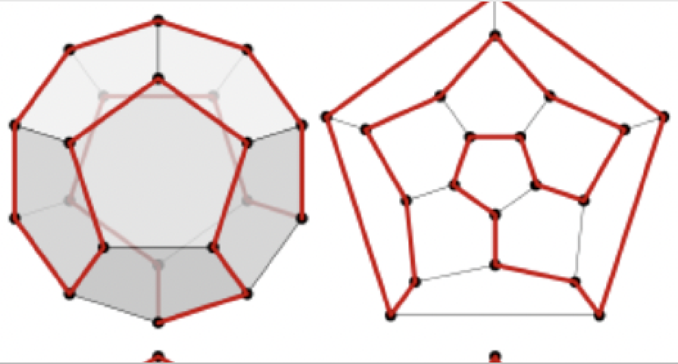


Wiliam Rowan Hamilton (1805 - 1865)

A trip around the world

Problem: Is it possible to walk through given cities visiting each city exactly once and returning to the place we started in?

Around the World



Hamiltonian Circuit

Definition

A **Hamiltonian circuit** is a simple closed path that passes through each vertex exactly once.

- A graph admits a Hamiltonian circuit if and only if it admits a polygon as a spanning subgraph.
- A graph is called **Hamiltonian** if it admits a Hamiltonian circuit.

The Hamiltonian Circuit Problem

Problem: For a given graph Γ decide whether it admits a Hamiltonian circuit.

- Note that this is a yes/no question.
- The yes answer can be justified via providing a specific sequence of vertices covering all the vertices of the graph in which any two consecutive vertices are adjacent (a certificate).
- The no answer is much harder to justify.
- For a specific graph (or family of graphs), the solution might be easy, however, solving the general problem with any finite graph as a possible input is one of the most (computationally) complex problems in Discrete Mathematics.
- Therefore, research in this area is focused on determining necessary or sufficient conditions (but not both necessary and sufficient) for the existence of a Hamiltonian circuit in a graph.
- This problem will be our focus for a while now.

A Crash Course in Computational Complexity

- A yes/no problem of size n (number of vertices in a graph, number of elements in a list, ...) for which the yes answer can be justified by providing a certificate whose validity can be verified in a number of steps that is polynomial in n is called an NP-**problem** (non-deterministic polynomial).
- If the set of possible certificates (not all of them valid) is exponential in n, one possible way to decide the problem is to check each possible certificate (in polynomial time).
- This algorithm is exponential in n, but the answer is definite.
- A yes/no decision problem of order n that can be decided via performing a number of steps polynomial in n is called a P-**problem** (polynomial).
- It is a famous open problem whether $P = NP$.

A Crash Course in Computational Complexity

- An NP-problem $\mathcal{P}$ is called NP-**complete** if every NP-problem can be answered by forming in polynomial time a corresponding problem in $\mathcal{P}$ whose yes/no answer is the same as the answer of the original NP-problem.
- Showing that a specific NP-problem $\mathcal{P}$ is NP-complete comes down to showing the polynomial reduction of all NP-problems to $\mathcal{P}$ **or** by polynomially reducing a known NP-complete problem to $\mathcal{P}$.
- There exists a long list of known NP-complete problems by now.
- Finding a polynomial time solution for any of the known NP-complete problems would mean that $P = NP$.
- The Hamiltonian Circuit Problem is NP-complete.

Computational Complexity for Beginners[1]

A GENTLE INTRODUCTION TO COMPUTATIONAL COMPLEXITY THEORY, AND A LITTLE BIT MORE

SEAN HOGAN

Abstract. We give the interested reader a gentle introduction to computational complexity theory, by providing and looking at the background leading up to a discussion of the complexity classes P and NP. We also introduce NP-complete problems, and prove the Cook-Levin theorem, which shows such problems exist. We hope to show that study of NP-complete problems is vital to countless algorithms that help form a bridge between theoretical and applied areas of computer science and mathematics.

Contents

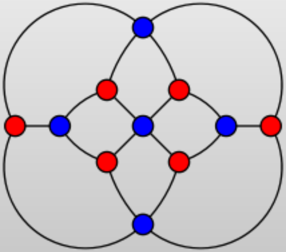
[1] available in *moodle*

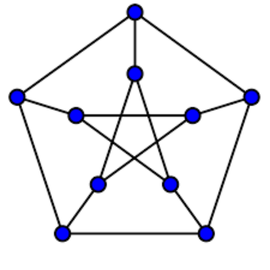
Some Problems for Discussion:

1. Design a schedule of bilateral discussions of five senators A, B, C, D, E that satisfies the following conditions:
   One of the participants of each discussion (except the last one) will attend the next discussion and no one will attend three consecutive discussions. Senator A wants to discuss with senators B, C, E; senator B with senators A, C, D, E; senator C with senators A, B, D, E; D needs to discuss with senators B, C and senator E with senators A, B, C. Represent the solution graphically.

Some Problems for Discussion:

1. Design a round trip of the city or explain why it is not possible.





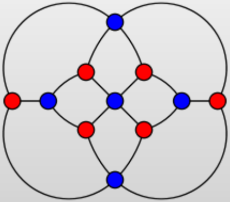
2. A mouse eats his way through a $3 \times 3 \times 3$ cube of cheese by tunnelling through all the 27 $1 \times 1 \times 1$ subcubes. If he starts in one corner and always moves onto an adjacent uneaten subcube, can he finish at the center of the cube? (Bondy, Murty: Graph theory with applications)

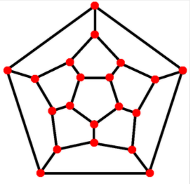
Some Problems for Discussion:

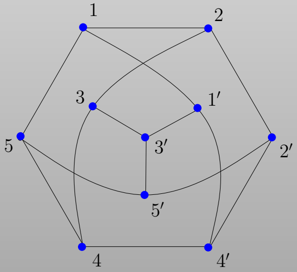
1. We have a standard deck of 52 cards, that is, each card has one of the values $A = 1$, 2, 3, 4, 5, 6, 7, 8, 9, 10, $J = 11$, $Q = 12$, $K = 13$ and one of the colors $\heartsuit, \diamondsuit, \clubsuit, \spadesuit$. Suppose that someone divided the cards into 13 piles of 4 cards. Show that it is possible to choose one card from each pile, so that the values are all distinct.

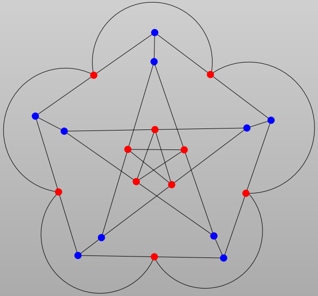
Hamiltonian Graphs

Some graphs instead of an introduction:





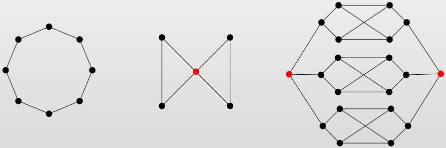




- **Hamiltonian path** in a graph G: a path that contains all vertices of G.
- **Hamiltonian graph**: graph that admits a Hamiltonian circuit.
- Parallel edges do not affect the existence of a circuit. For a graph on more than one vertex and with a loop, a Hamiltonian circuit does not contain this loop. Thus we will consider graphs without loops and parallel edges.
- Hamiltonian graphs:
  - cycles
  - complete graphs on at least 3 vertices
  - complete bipartite graphs with parts of even order $n \geq 2$
  - cubes of dimension $n \geq 2$
  - ...
- Non-Hamiltonian graphs:
  - disconnected graphs
  - trees
  - bipartite graphs on an odd number of vertices (do they need to be complete bipartite graphs?)
  - ...

Necessary Conditions

A Hamiltonian graph clearly needs to be connected. Is it possible to say something more? Let's take a look at these three graphs:



- The cycle is Hamiltonian. What about the other two graphs?
- Omit a vertex from a Hamiltonian graph, that is, from a Hamiltonian circuit. The result is a graph with a Hamiltonian path. Equivalently, a Hamiltonian graph is 2-connected (i.e., one needs to remove at least two vertices to make it disconnected). Therefore the butterfly does not admit a Hamiltonian circuit.
- Omit two vertices from a Hamiltonian graph, that is, from a Hamiltonian circuit. The resulting graph has at most two components. Therefore also graph on the right does not admit a Hamiltonian circuit, even though it is 2-connected.

Seminár z teórie grafov

Srdečne Vás pozývame na seminár z teórie grafov, ktorý sa uskutoční vo štvrtok 03. 10. 2024 o 09:50 na FMFI UK v miestnosti M-213.

Prednášajúci/speaker: **Andre Raspaud**, LABRI, Bordeaux University, France

Názov/Title: **Induced 2-improper edge coloring**

Algebraic Graph Theory Seminar

Time: October 4th 2024 (Friday), 13:15 CEST

Place: Room M-VIII and MS Teams

Speaker: **Eze Leonard Chidiebere** (Comenius University)

Title: **Theoretical nad Computational Approaches to Determining Sets of Orders of (k, g)-Graphs**