Graph Theory
Hamiltonian circuits II

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17.10.2024

Course Info

- previous lectures are posted in *moodle*
- the first homework is posted on *moodle*; it is due on Sunday, Oct. 27, at midnight
- start early, so that you have a chance to ask questions

Hamiltonian circuit

Definícia

A **Hamiltonian circuit** is a simple closed path that passes through each vertex exactly once.

- A graph admits a Hamiltonian circuit if and only if it admits a polygon as a spanning subgraph.
- A graph is called **Hamiltonian** if it admits a Hamiltonian circuit.

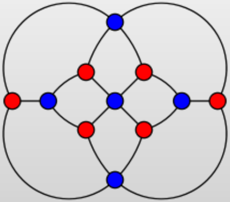
Hamiltonian graphs

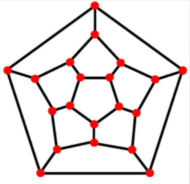
Problem: Given a graph G, determine whether it contains a Hamiltonian circuit.

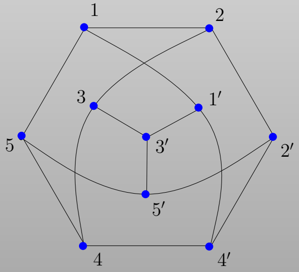
- For a specific graph, or even for an infinite class of graphs, this decision problem can be computationally easy; the general problem, however, is one of the more computationally complex problems in Discrete Mathematics. It belongs to the class of problems deemed NP-complete.
- This is why finding necessary and sufficient conditions for a graph to be Hamiltonian which can be tested in a meaningful time is not feasible, and research focuses on finding necessary or sufficient conditions for the existence of a Hamiltonian circuit.
- This will be our focus for a while.

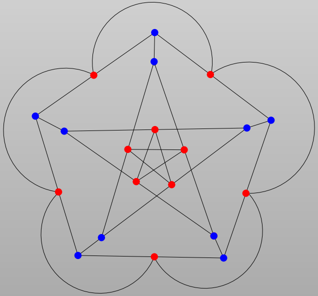
Hamiltonian graphs (?)

A few graphs instead of an introduction:









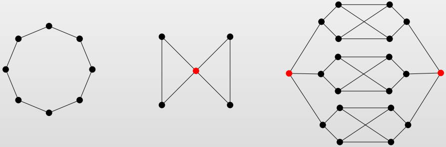
- A **Hamiltonian path** in a graph G: a path that visits all the vertices of the graph (exactly once)
- A **Hamiltonian graph**: a graph, which contains a Hamiltonian circuit
- Every graph that contains a Hamiltonian circuit, contains a Hamiltonian path, but not every graph that contains a Hamiltonian path, contains a Hamiltonian circuit
- The graphs we shall consider will contain no loops or parallel edges (as those cannot be included in a Hamiltonian circuit)
- Hamiltonian graphs:
  - cycles
  - complete graphs K_n on at least 3 vertices
  - complete bipartite graphs $K_{m,n}$ in which both parts are of even orders
  - n-dimensional cubes, $n \geq 2$
  - ...
- Nonhamiltonian graphs:
  - disconnected graphs
  - trees
  - bipartite graphs of odd orders
  - ...

Necessary conditions

- A Hamiltonian graphs is necessarily connected. What else can we say for sure?
- Every vertex in a Hamiltonian graph of order at least 2 must be of degree at least 2.

Necessary conditions

Let us consider the following three graphs:



- A cycle is Hamiltonian. How about the other two?
- A 1-connected graph is one where erasing a vertex will lead to a disconnected graph. A vertex like that is called an **articulation**. A 1-connected graph is clearly not Hamiltonian.
- This means that a Hamiltonian graph must be at least 2-connected; that is why the butterfly graph is not Hamiltonian.
- Erasing 2 vertices from a graph (this means, erasing 2 vertices from a Hamiltonian circuit) will result in a graph with *at most* two components. This is why the last graph on the very right is also not Hamiltonian even though it is 2-connected.

Necessary conditions

Veta

Let G be Hamiltonian. Then, for each $\emptyset \neq S \subset V(G)$, the number of connected components $c(G \setminus S)$, of the induced graph $G \setminus S$ does not exceed $|S|$.

Proof: Let C be a Hamiltonian circuit in G. All the vertices of G belong to C, therefore

$$c(G \setminus S) \leq c(C \setminus S).$$

It suffices to show that

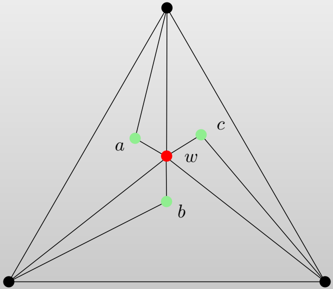
$$c(C \setminus S) \leq |S|.$$

That is obvious, as the components of $C \setminus S$ are paths (a detailed proof could be done using induction on $|S|$). □

Corollary

Complete bipartite graph is $K_{m,n}$ is Hamiltonian only if $m = n$.

However, there exist graphs that satisfy the conditions from the previous theorem, and are still not Hamiltonian. I.e., the condition stated in the theorem is necessary but not sufficient.



Obr.: Non-Hamiltonian graph that satisfies the condition from the previous theorem.

- Verify that it satisfies the condition
- Argue that it is non-Hamiltonian.

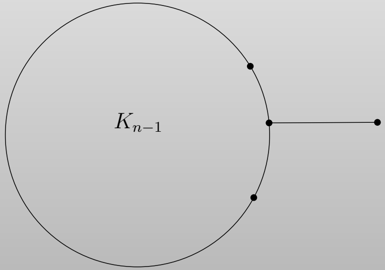
Sufficient conditions

Three basic types of conditions:

- **Degree-sequence requirements** are among the most classical.
- They start off the intuitive assumption that the more edges in a graph, the more likely the graph contains a Hamiltonian circuit.
- The main question is however: *What does it mean 'A graph has a lot of edges'?*

Degree-sequence requirements

- Assuming that a graph contains 'almost all' possible edges is not enough. This can be seen by considering the complete graph K_n with a pendant edge added; which contains $\frac{n(n-1)}{2} + 1$ of the $\frac{n(n+1)}{2}$ possible edges



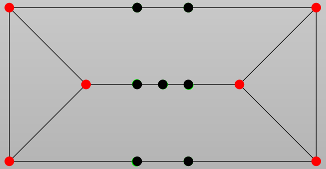
Obr.: Non-Hamiltonian graph that contains almost all possible edges

Degree-sequence requirements

- This means that there exist graphs that contain almost all possible edges and are still not Hamiltonian. Therefore, it also appears important how the edges are distributed among the edges, i.e., the degree sequence appears relevant.

Sufficient conditions

- **Structural conditions**. For example, conditions that prohibit induced subgraphs from a specific set of graphs $\mathcal{F}$. A prominent role is played by the star graph $K_{1,3}$.
- Well chosen prohibited subgraphs together with additional requirements (guaranteeing, for example, the 2-connectedness; which as we know, may not suffice) may guarantee the existence of a Hamiltonian circuit. The main question is of course: *Which are the additional requirements and which graphs should be prohibited?*
- This is a newer research direction going back to the 1970's.



Obr.: Non-Hamiltonian 2-connected graph without an induced $K_{1,3}$

Sufficient conditions

- **Recursive generation of Hamiltonian graphs using graph operations** such as graph products.
- Research focuses on various types of graph products (direct, lexicographic, ...) with possible modification as well as on constructions such as line graph[1]
- For example, the line graph of an Eulerian graph is Hamiltonian (Proof?)
- Similarly, the line graph of a Hamiltonian graph is Hamiltonian (Proof?)

[1] The vertices of the **line graph** $L(G)$ of a graph $G = (V, E)$ are the edges of G with two of them adjacent if and only if they shared a vertex in G.

Sufficient conditions

- One can also consider a reversed order of doing things. Choose a class of graphs $\mathcal{G}$ together with a graph operation or operations. Find a criterion or an algorithm that will determine whether a graph has been formed by the selected operation(s), and in case of positive answer, it will also provide the graphs from which our graph has been constructed. Then determine the *root graphs* of these constructions, i.e., graphs that cannot be constructed from the smaller graphs, and which allow one to construct iteratively all the graphs in the class $\mathcal{G}$.
- Having done all the above, one can ask which circumstances (which root graphs and which sequences of operations) guarantee the hamiltonicity of the constructed graph.

We will focus on requirements of the first and second type.

Sufficient conditions for the degree-sequence

Veta (A. Dirac, 1952)

A graph on $n \geq 3$ vertices of minimal degree $\delta \geq n/2$ is Hamiltonian.

Dôkaz.

- Kompletný graf K_2 nie je hamiltonovský a každý vrchol má stupeň 1. Preto je podmienka $n \geq 3$ potrebná. Na druhej strane, K_n je očividne hamiltonovský pre všetky $n \geq 3$.
- Ďalej postupujme nepriamo. Nech G vyhovuje predpokladom vety a nie je hamiltonovský. Potom nie je kompletný a má aspoň dva nesusedné vrcholy.
- G vnoríme do *maximálneho* nehamiltonovského grafu G^*: Zvolíme nehranu $e = uv \notin G$.
  - Ak $G + e$ je hamiltonovský, hranu e vymažeme.
  - Ak $G + e$ nie je hamiltonovský, hranu e ponecháme.
  - Iterujeme.

Dôkaz Diracovej vety, pokračovanie

- Po konečnom počte krokov dostaneme graf, pomenujme ho G^*, s $\delta \geq n/2$, ktorý nie je hamiltonovský a pridanie ľubovoľnej hrany $uv \notin E(G^*)$ spôsobí vznik hamiltonovskej kružnice.
- Preto G^* obsahuje hamiltonovskú cestu

$$P = u, u_2, u_3, \ldots, u_{n-2}, u_{n-1}, v$$

  z u do v (pre ľubovoľnú dvojicu nesusedných vrcholov u, v).
- Položme $S = \{i : uu_{i+1} \in E(G^*)\}, T = \{i : vu_i \in E(G^*)\}$, $1 \leq i \leq n-1$.
- Platí

$$|S| = \deg_{G^*} u \geq \deg_G u \geq \frac{n}{2}, \quad |T| = \deg_{G^*} v \geq \deg_G v \geq \frac{n}{2}$$

Dôkaz Diracovej vety, pokračovanie

- Pripomeňme si jednoduchú rovnosť

$$|S| + |T| = |S \cup T| + |S \cap T|, \qquad (1)$$

kde podľa predpokladu $|S| + |T| \geq \frac{n}{2} + \frac{n}{2} = n$, zatiaľ čo $S \cup T \subseteq \{1, 2, \ldots, n-1\}$, a teda $|S \cup T| \leq n - 1$.

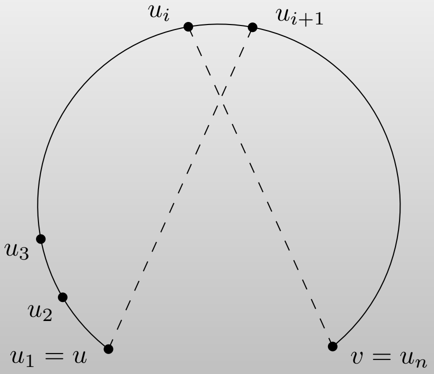
- To znamená, že $|S \cap T| > 0$ a teda, že $S \cap T$ je neprázdne.
- A teda existuje $i \in S \cap T$, kde u_{i+1} je susedom u a u_i je susedom v, pričom u_i a u_{i+1} sú susedia, lebo nasledujú po sebe na hamiltonovskej ceste P
- Potom ale

$$u, u_2, \ldots, u_i, v, u_{n-1}, \ldots u_{i+1}, u$$

je hamiltonovská kružnica v G^* - spor s voľbou G^*.

□

Postačujúce podmienky na stupne



Obr.: Hamiltonovská cesta v $G^\star$ spolu s vynútenými hranami

Do not forget about the homework.