Graph Theory
Lecture #8
Graph Automorphisms
Groups of Automorphisms and Group Automorphisms
Highly symmetric graphs
II.

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November 14, 2024

Automorphisms of Graphs

- ⋆ An isomorphism in $Iso(\Gamma, \Gamma)$ is called a **(graph) automorphism of** Γ; i.e., a graph automorphism of Γ is a bijection $\varphi : V(\Gamma) \to V(\Gamma)$ satisfying the property

$$u \sim v \iff \varphi(u) \sim \varphi(v)$$

- ⋆ The set of all automorphisms of Γ together with the operation of composition is called the **automorphism group of** Γ, denoted $Aut(\Gamma)$.
- ⋆ It is indeed a group:
  - the composition of two automorphisms is an automorphism
  - the operation of composition is associative
  - the neutral element of composition is the bijection $id_{V(\Gamma)} : V(\Gamma) \to V(\Gamma)$,

$$id_{V(\Gamma)}(u) = u, \text{ for all } u \in V(\Gamma)$$

  - the inverse of an automorphism is an automorphism

Automorphism Groups of Small Graphs

https://www.graphclasses.org/smallgraphs.html

Information System on Graph Classes and their Inclusions

Global | ISGCI home | Java | All classes | References | ✉ | Find class

Contents

- Graphs ordered alphabetically
- Graphs ordered by number of vertices
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  - 13 vertices
- Configurations XC
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- Families XF
- General families
- More graphs in the *House of Graphs* (external link)

Automorphism Groups of Graphs of Order 2

2 vertices - Graphs are ordered by increasing number of edges in the left column. The list contains all 2 graphs with 2 vertices.

$2K_1$ g6: A?

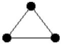
K_2 g6: A_

Automorphism Groups of Graphs of Order 3

3 vertices - Graphs are ordered by increasing number of edges in the left column. The list contains all 4 graphs with 3 vertices.

$3K_1$ = co-triangle g6: B?

triangle = K_3 = C_3 g6: Bw



$\overline{P_3}$ g6: BO

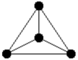
P_3 g6: Bg

Automorphism Groups of Graphs of Order 4

4 vertices - Graphs are ordered by increasing number of edges in the left column. The list contains all 11 graphs with 4 vertices.

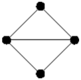
$4K_1 = \overline{K_4}$ g6: C?

$K_4 = W_3$ g6: C~



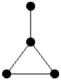
co-diamond g6: CC

diamond = K_4 - e = 2-fan g6: Cz



co-paw g6: CE

paw = 3-pan g6: Cx



$2K_2 = \overline{C_4}$ g6: CK

$C_4 = K_{2,2}$ g6: Cr

General Results about Automorphism Groups of Graphs

Theorem

1. $$Aut(\Gamma) = Aut(\overline{\Gamma})$$
2. $$Aut(K_n) = \mathbb{S}_n$$
3. $$Aut(C_n) = \mathbb{D}_n$$
4. *If* $\Gamma_1 \ncong \Gamma_2$, *then* $Aut(\Gamma_1) \times Aut(\Gamma_2) \leq Aut(\Gamma_1 \dot{\cup} \Gamma_2)$.
5. *If* $\Gamma_1 \cong \Gamma_2$, *then* $Aut(\Gamma_1) \times Aut(\Gamma_2) \times \mathbb{Z}_2 \leq Aut(\Gamma_1 \dot{\cup} \Gamma_2)$.

A Connection to Group Theory

- ⋆ A **group isomorphism** between groups $(G_1, *)$ and $(G_2, \cdot)$ is a bijection $\varphi : G_1 \to G_2$ 'preserving the group structure':

$$\varphi(a * b) = \varphi(a) \cdot \varphi(b), \text{ for all } a, b \in G_1$$

- ⋆ A group isomorphism $\varphi : G \to G$ is called a **group automorphism**
- ⋆ The set of all automorphisms of G together with the operation of composition is called the **automorphism group of** G, denoted $Aut(G)$.
- ⋆ It is indeed a group:
  - the composition of two automorphisms is an automorphism
  - the operation of composition is associative
  - the neutral element of composition is the bijection $id_G : G \to G$,

$$id_G(g) = g, \text{ for all } u \in G$$

  - the inverse of an automorphism is an automorphism

Group Isomorphism Problem

Input: Two groups $(G_1, *), (G_2, \cdot)$
Question: Does there exist an isomorphism between $(G_1, *)$ and $(G_2, \cdot)$?
In case of a positive answer, the algorithm should provide a specific isomorphism.

Group Isomorphism Problem vs. Graph Isomorphism Problem

The Graph Isomorphism and Group Isomorphism Problems appear to be of similar complexity:

- ⋆ Both can be answered via visiting all bijections between the two objects
- ⋆ The verification for a specific bijection involves visiting all pairs

Group Isomorphism Problem vs. Graph Isomorphism Problem

Construction. Let K be a finite group.

Define a graph $\Gamma(K)$ with vertex set $K \times K$ and edges:

$$(a,b) \sim (c,d) \iff a = c \vee b = d \vee ab = cd.$$

Theorem $K_1 \cong K_2 \iff \Gamma(K_1) \cong \Gamma(K_2)$.

Exercise. Prove that $\Gamma(K)$ is a Cayley graph over $K \times K$.

Exercise. Prove that $\Gamma(\mathbb{Z}_4) \not\cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

$\mathbb{Z}_4 \rightarrow$

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

$\mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow$

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

Automorphism Groups of Groups

Problem 1: Determine the automorphism group of $(Z_n, \oplus)$, $n \geq 2$

Problem 2: Determine the automorphism group of $(Z_n^{\#}, \odot)$, $n \geq 2$

Strongly Regular Graphs

A k-regular graph Γ of order v is said to be a (v, k, λ, μ)-**strongly regular graph** if

1. the number of common neighbors for any pair of adjacent vertices u, v of Γ is equal to λ
2. the number of common neighbors for any pair of non-adjacent vertices u, v of Γ is equal to μ

Examples:

- $\star$ complete graphs K_n, $n \geq 2$
- $\star$ the complement of a (v, k, λ, μ)-strongly regular graph (parameters?)
- $\star$ the Petersen graph

Strongly Regular Graphs

- $\star$ **Johnson graphs**, $J(n,k)$; $V = \binom{n}{k}$,
  $\mathcal{S} \sim \mathcal{T}$ iff $|\mathcal{S} \cap \mathcal{T}| = k-1$ (parameters?)
- $\star$ **Kneser graphs**, $K(n,k)$; $V = \binom{n}{k}$,
  $\mathcal{S} \sim \mathcal{T}$ iff $\mathcal{S} \cap \mathcal{T} = \emptyset$ (parameters?)
- $\star$ $J(n,2) \cong \overline{K(n,2)}$
- $\star$ $J(n,2)$ is the line graph for K_n;
- $\star$ Petersen $\cong K(5,2)$
- $\star$ **Latin square graphs**; V consists of the windows of a Latin square, and two windows are connected if they share the column, the row, or the contents;
  graphs $\Gamma(G)$ defined for groups G belong to this class, but not all Latin square graphs are $\Gamma(G)$ graphs for some G

The Stabilizer of a Vertex in a Group of Permutations

Definition

Let $G \leq \mathbb{S}_X$, and $x \in X$. The **stabilizer of** x in G is the set of all permutations in G that fix (stabilize) x; i.e.,

$$Stab_G(x) = \{\varphi \in G \mid \varphi(x) = x\}$$

- ⋆ $Stab_G(x)$ is a subgroup of G for every $x \in X$
- ⋆ if there exists a permutation $\varphi \in G$ such that $\varphi(x) = y$, for some $x, y \in X$, then $|Stab_G(x)| = |Stab_G(y)|$, and $Stab_G(x) = \varphi Stab_G(y) \varphi^{-1}$
- ⋆ The proof follows from the easy observations $Stab_G(x) \leq \varphi Stab_G(y) \varphi^{-1}$ and $\varphi Stab_G(y) \varphi^{-1} \leq Stab_G(x)$

The Stabilizer of a Vertex in a Graph

Definition

Let $\Gamma = (V, E)$ be a graph, $G = Aut(\Gamma)$, and $u \in V$. The **stabilizer of** u in G is the set of all automorphisms of Γ that fix (stabilize) u; i.e.,

$$Stab_G(u) = \{\varphi \in G \mid \varphi(u) = u\}$$

- ⋆ all of these concepts are 'specifications' of the general results on groups of permutations to groups of permutations that are also graph automorphisms
- ⋆ $Stab_G(u)$ is a subgroup of G for every $u \in V$
- ⋆ if there exists an automorphism $\varphi \in G$ such that $\varphi(u) = v$, for some $u, v, \in V$, then $|Stab_G(u)| = |Stab_G(v)|$, and $Stab_G(u) = \varphi Stab_G(v) \varphi^{-1}$

The Stabilizer of a Set

Definition

1. Let $\Gamma = (V, E)$ be a graph, $G = Aut(\Gamma)$, and $S \subseteq V$. The **pointwise stabilizer of** S in G is the set of all automorphisms of Γ that fix (stabilize) every $u \in S$; i.e.,

$$Stab_G(S) = \{\varphi \in G \mid \varphi(u) = u, \text{ for all } u \in S\}$$

2. The **setwise stabilizer of** S in G is the set of all automorphisms of Γ that fix (stabilize) the set S (but may mix the elements in S); i.e.,

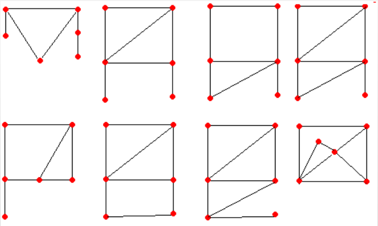
$$Stab_G(\{S\}) = \{\varphi \in G \mid \varphi(u) \in S, \text{ for all } u \in S\}$$

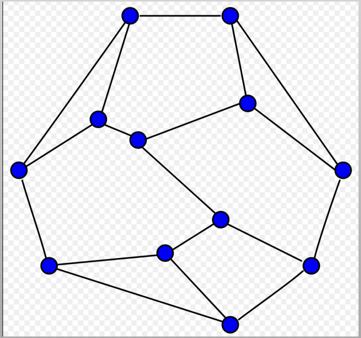
- ⋆ $Stab_G(S) \leq Stab_G(\{S\}) \leq G$, for every $S \subseteq V$
- ⋆ $Stab_G(S) = \bigcap_{u \in S} Stab_G(u)$
- ⋆ all the above definitions and observations clearly extend to general groups of permutations $G \leq \mathbb{S}_X$

Rigid (Asymmetric) Graphs

Problem 1:
Find a graph Γ of order ≥ 2 that has a trivial automorphism group; i.e., $Aut(\Gamma) = \{id_{V(\Gamma)}\}$.
Problem 2:
Find a graph Γ of the smallest order that has a trivial automorphism group; i.e., $Aut(\Gamma) = \{id_{V(\Gamma)}\}$.
Problem 3:
Find a cubic graph Γ of the smallest order that has a trivial automorphism group; i.e., $Aut(\Gamma) = \{id_{V(\Gamma)}\}$.
Problem 4:
Find a planar graph Γ of the smallest order that has a trivial automorphism group; i.e., $Aut(\Gamma) = \{id_{V(\Gamma)}\}$.

Rigid (Asymmetric) Graphs





(Rigid) Asymmetric Graphs

A graph Γ is called **asymmetric** if it does not have a non-trivial automorphism.

$$\Leftrightarrow$$

$$Aut(\Gamma) \text{ is trivial.}$$

Theorem (Erdős, Renyi (1963))

Almost all graphs are asymmetric.

i.e.,

$$\lim_{n\to\infty} \frac{\#\text{ of asymmetric graphs of order } \le n}{\#\text{ graphs of order } \le n} = 1$$

Problem:
Is there a strongly regular graph of order ≥ 2 with a trivial automorphism group?

Asymmetric Graphs

Erdös, Rényi:

- ⋆ Symmetrization: removing (r) and adding (s) edges to make a graph symmetric
- ⋆ Degree of asymmetry $A(\Gamma)$: the minimum of $r + s$ taken over all possible symmetrizations
- ⋆ The asymmetry of a graph of order n can not exceed $\frac{n-1}{2}$; and this estimate is asymptotically best possible
- ⋆ The relative asymmetry of Γ,

$$a(\Gamma) = \frac{A(\Gamma)}{\frac{n-1}{2}}$$

$0 \leq a(\Gamma) \leq 1$

Symmetry vs. Asymmetry: Minimal Asymmetric Graphs

An undirected graph G on at least two vertices is **minimal asymmetric** if G is asymmetric and no proper induced subgraph of G on at least two vertices is asymmetric.

Minimal Asymmetric Graphs

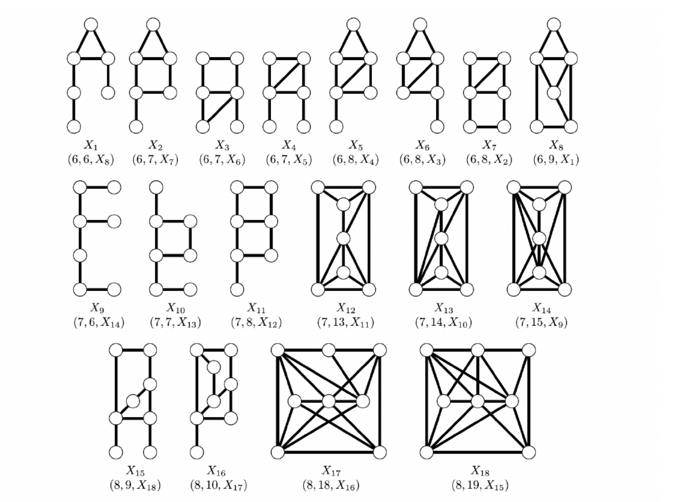
Theorem (Schweitzer, Pascal; Schweitzer, Patrick, 2017)

There are exactly 18 finite minimal asymmetric undirected graphs up to isomorphism.

Nešetřil's conjecture: There are exactly 18 minimal asymmetric graphs (coming in 9 complementary pairs).

Nešetřil and Sabidussi earlier established a close connection between minimal asymmetric graphs and minimal involution-free graphs.

Minimal Asymmetric Graphs



Orbit of a Permutation Group

- ⋆ Recall the cycle decomposition of a permutation $\varphi \in \mathbb{S}_n$ into disjoint cycles.
- ⋆ Note that the cycles (including the single element cycles which we usually do not list) form a *partition* of the set $\{1, 2, \ldots, n\}$
- ⋆ also note that any two elements i and j in $\{1, 2, \ldots, n\}$ belong to the same cycle if and only if there exists a $0 \leq k \leq |\varphi| - 1$ such that $\varphi^k(i) = j$

These observations generalize into the following:

Definition

Let G be a subgroup of the full symmetric group $\mathbb{S}_X$. The **orbit** of G containing the element $x \in X$ is the set of all elements $y \in X$ such that there exists and element $\varphi \in G$ such that $\varphi(x) = y$.

Orbits of a Permutation Group

Theorem

1. *If $x, y \in X$ belong to the same orbit of $G \leq \mathbb{S}_X$ on X, their stabilizers are conjugates in G*[1].
2. *If $G \leq \mathbb{S}_X$, the orbits of G on X form a partition of X.*
3. *The relation on X defined for any pair of elements $x, y \in X$ via the rule*

$$x \sim y \text{ iff } \exists \varphi \in G \text{ such that } \varphi(x) = y$$

   is an equivalence relation on X
4. *The cycles of φ are the orbits of $\langle \varphi \rangle$.*

[1] Is this an if and only if statement?

Orbit of a Permutation Group

Note that a graph Γ has a trivial automorphism group if and only each vertex of Γ is its own orbit, i.e., if Γ has a non-trivial automorphism group, it has at least one orbit of size > 1.