Graph Theory
Lecture #9
Graph Automorphisms
and
Automorphism Groups of Graphs

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Strongly Regular Graphs

A k-regular graph Γ of order v is said to be a (v, k, λ, μ)-**strongly regular graph** if

1. the number of common neighbors for any pair of adjacent vertices u, v of Γ is equal to λ
2. the number of common neighbors for any pair of non-adjacent vertices u, v of Γ is equal to μ

Examples:

- $\star$ complete graphs K_n, $n \geq 2$
- $\star$ the complement of a (v, k, λ, μ)-strongly regular graph (parameters?)
- $\star$ the Petersen graph

Strongly Regular Graphs

- **Johnson graphs**, $J(n,k)$; $V = \binom{n}{k}$,
  $\mathcal{S} \sim \mathcal{T}$ iff $|\mathcal{S} \cap \mathcal{T}| = k - 1$ (parameters?)
- **Kneser graphs**, $K(n,k)$; $V = \binom{n}{k}$,
  $\mathcal{S} \sim \mathcal{T}$ iff $\mathcal{S} \cap \mathcal{T} = \emptyset$ (parameters?)
- $J(n,2) \cong \overline{K(n,2)}$
- $J(n,2)$ is the line graph for K_n;
- Petersen $\cong K(5,2)$
- **Latin square graphs**; V consists of the windows of a Latin square, and two windows are connected if they share the column, the row, or the contents;
  graphs $\Gamma(G)$ defined for groups G belong to this class, but not all Latin square graphs are $\Gamma(G)$ graphs for some G

Back to the automorphism groups of graphs

But first, we start talking about a general concept that concerns all groups of permutations (not just the automorphism groups of graphs)

The Stabilizer of a Vertex in a Group of Permutations

Definition

Let $G \leq \mathbb{S}_X$ (i.e., let G be a group of permutations of a set X), and let $x \in X$. The **stabilizer of** x in G is the set of all permutations in G that fix (stabilize) x; i.e.,

$$Stab_G(x) = \{\varphi \in G \mid \varphi(x) = x\}$$

- ⋆ $Stab_G(x)$ is a subgroup of G for every $x \in X$
- ⋆ if there exists a permutation $\varphi \in G$ such that $\varphi(x) = y$, for some $x, y \in X$, then $|Stab_G(x)| = |Stab_G(y)|$, and $Stab_G(x) = \varphi Stab_G(y) \varphi^{-1}$
- ⋆ The proof follows from the easy observations $Stab_G(x) \leq \varphi Stab_G(y) \varphi^{-1}$ and $\varphi Stab_G(y) \varphi^{-1} \leq Stab_G(x)$

The Stabilizer of a Vertex in a Graph

Definition

Let $\Gamma = (V, E)$ be a graph, $G = Aut(\Gamma)$, and $u \in V$. The **stabilizer of** u in G is the set of all automorphisms of Γ that fix (stabilize) u; i.e.,

$$Stab_G(u) = \{\varphi \in G \mid \varphi(u) = u\}$$

- ⋆ all of these concepts are 'specifications' of the general results on groups of permutations to groups of permutations that are also graph automorphisms
- ⋆ $Stab_G(u)$ is a subgroup of G for every $u \in V$
- ⋆ if there exists an automorphism $\varphi \in G$ such that $\varphi(u) = v$, for some $u, v, \in V$, then $|Stab_G(u)| = |Stab_G(v)|$, and $Stab_G(u) = \varphi Stab_G(v) \varphi^{-1}$

The Stabilizer of a Set of Vertices in a Graph

Definition

1. Let $\Gamma = (V, E)$ be a graph, $G = Aut(\Gamma)$, and $S \subseteq V$. The **pointwise stabilizer of** S in G is the set of all automorphisms of Γ that fix (stabilize) every $u \in S$; i.e.,

$$Stab_G(S) = \{\varphi \in G \mid \varphi(u) = u, \text{ for all } u \in S\}$$

2. The **setwise stabilizer of** S in G is the set of all automorphisms of Γ that fix (stabilize) the set S (but may mix the elements in S); i.e.,

$$Stab_G(\{S\}) = \{\varphi \in G \mid \varphi(u) \in S, \text{ for all } u \in S\}$$

- $\star$ $Stab_G(S) \leq Stab_G(\{S\}) \leq G$, for every $S \subseteq V$
- $\star$ $Stab_G(S) = \bigcap_{u \in S} Stab_G(u)$

The Stabilizer of a Set

All the above definitions and observations extend to general groups of permutations $G \le \mathbb{S}_X$:

Definition

1. Let $G \le \mathbb{S}_X$, and let $S \subseteq X$. The **pointwise stabilizer of** S in G is the set of all permutations in G that fix (stabilize) every $u \in S$; i.e.,

$$Stab_G(S) = \{\varphi \in G \mid \varphi(u) = u, \text{ for all } u \in S\}$$

2. The **setwise stabilizer of** S in G is the set of all permutations in G that fix (stabilize) the set S (but may mix the elements in S); i.e.,

$$Stab_G(\{S\}) = \{\varphi \in G \mid \varphi(u) \in S, \text{ for all } u \in S\}$$

And now for another but related topic ...

Rigid (Asymmetric) Graphs

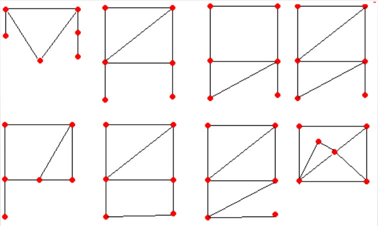
Problem 1:
Find a graph Γ of order ≥ 2 that has a trivial automorphism group; i.e., $Aut(\Gamma) = \{id_{V(\Gamma)}\}$, i.e., the only permutation of the vertices in $V(\Gamma)$ that is actually a graph automorphism (preserves edges and non-edges) is the trivial permutation $id_{V(\Gamma)} : u \mapsto u$, for all $u \in V(\Gamma)$ **(which is always an automorphism)**

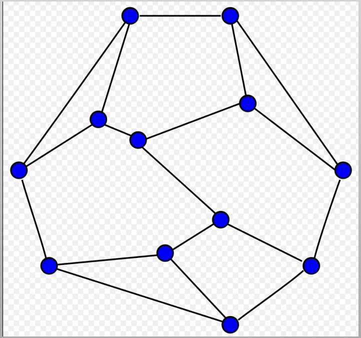
Problem 2:
Find a graph Γ **of the smallest order** that has a trivial automorphism group.

Problem 3:
Find a cubic graph Γ **of the smallest order** that has a trivial automorphism group.

Problem 4:
Find a planar graph Γ **of the smallest order** that has a trivial automorphism group.

Rigid (Asymmetric) Graphs





(Rigid) Asymmetric Graphs

A graph Γ is called **asymmetric** if it does not have a non-trivial automorphism.

$$\Leftrightarrow$$

$$Aut(\Gamma) \text{ is trivial.}$$

Theorem (Erdős, Renyi (1963))

Almost all graphs are asymmetric.

i.e.,

$$\lim_{n\to\infty} \frac{\#\text{ of asymmetric graphs of order } \leq n}{\#\text{ graphs of order } \leq n} = 1$$

Problem:
Is there a strongly regular graph of order ≥ 2 with a trivial automorphism group?
(We are asking about the existence, not even about a smallest such graph.)

Asymmetric Graphs

Erdös, Rényi:

- ⋆ Symmetrization: removing (r) and adding (s) edges to make a graph symmetric
- ⋆ Degree of asymmetry $A(\Gamma)$: the minimum of $r + s$ taken over all possible symmetrizations
- ⋆ The asymmetry of a graph of order n can not exceed $\frac{n-1}{2}$; and this estimate is asymptotically best possible
- ⋆ The relative asymmetry of Γ,

$$a(\Gamma) = \frac{A(\Gamma)}{\frac{n-1}{2}}$$

$0 \leq a(\Gamma) \leq 1$

Symmetry vs. Asymmetry: Minimal Asymmetric Graphs

An undirected graph G on at least two vertices is **minimal asymmetric** if G is asymmetric and no proper induced subgraph of G on at least two vertices is asymmetric.

Minimal Asymmetric Graphs

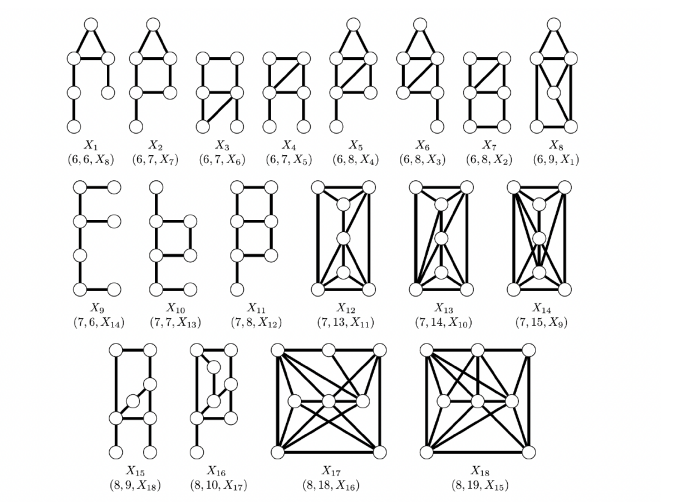
Theorem (Schweitzer, Pascal; Schweitzer, Patrick, 2017)

There are exactly 18 finite minimal asymmetric undirected graphs up to isomorphism.

Nešetřil's conjecture (preceding the theorem): There are exactly 18 minimal asymmetric graphs (coming in 9 complementary pairs).

Nešetřil and Sabidussi earlier established a close connection between minimal asymmetric graphs and minimal involution-free graphs.

Minimal Asymmetric Graphs



And now another fundamental concept concerning groups of permutations and groups of graph automorphisms ...

Orbit of a Permutation Group

- $\star$ Recall the cycle decomposition of a permutation $\varphi \in \mathbb{S}_n$ into disjoint cycles.
- $\star$ Note that the cycles (including the single element cycles which we usually do not list) form a *partition* of the set $\{1, 2, \ldots, n\}$
- $\star$ also note that any two elements i and j in $\{1, 2, \ldots, n\}$ belong to the same cycle if and only if there exists a $0 \leq k \leq |\varphi| - 1$ such that $\varphi^k(i) = j$

These observations generalize into the following:

Definition

Let G be a subgroup of the full symmetric group $\mathbb{S}_X$. The **orbit** of G containing the element $x \in X$ is the set of all elements $y \in X$ such that there exists and element $\varphi \in G$ such that $\varphi(x) = y$.

Orbits of a Permutation Group

Theorem

1. *If $x, y \in X$ belong to the same orbit of $G \leq \mathbb{S}_X$ on X, their stabilizers are conjugates in G*[1].
2. *If $G \leq \mathbb{S}_X$, the orbits of G on X form a partition of X.*
3. *The relation on X defined for any pair of elements $x, y \in X$ via the rule*

$$x \sim y \text{ iff } \exists \varphi \in G \text{ such that } \varphi(x) = y$$

   is an equivalence relation on X
4. *The cycles of φ are the orbits of $\langle \varphi \rangle$.*

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[1] Is this an if and only if statement?

Lemma

Let $G \leq \mathbb{S}_X$ and let $\mathcal{O}$ be an orbit of G. Then,

$$|G| = |Stab_G(x)| \cdot |\mathcal{O}|,$$

for every $x \in \mathcal{O}$.

Proof: Let $\mathcal{O} = \{x = x_1, x_2, \ldots, x_{|\mathcal{O}|}\}$ and $\varphi_1, \varphi_2, \ldots, \varphi_{|\mathcal{O}|} \in G$ such that $\varphi_i(x_1) = x_i$. Then

$$G = \bigcup_{1 \leq i \leq |\mathcal{O}|} \varphi_i Stab_G(x_1)$$

$\square$

Orbit of an Automorphism Group of a Graph

- ⋆ Orbits with respect to the automorphism group of a graph Γ form a partition of the vertex set $V(\Gamma)$
- ⋆ For any two vertices $u, v \in G(\Gamma)$ that belong to the same orbit, there exists an automorphism $\varphi \in Aut(\Gamma)$ mapping u to v, $\varphi(u) = v$
- ⋆ If u and w in $V(\Gamma)$ do not belong to the same orbit, there is **no** automorphism $\varphi \in Aut(\Gamma)$ mapping u to v
- ⋆ a graph Γ has a trivial automorphism group if and only each vertex of Γ is its own orbit, i.e., if Γ has a non-trivial automorphism group, it has at least one orbit of size > 1.
- ⋆ What are the orbits of K_n, $K_{m,n}$, C_n and a star?

Vertex-Transitive Graphs

Definition

A graph $\Gamma = (V, E)$ is called **vertex-transitive** if for every pair $u, v \in V$ there exists an automorphism $\varphi \in Aut(\Gamma)$ such that $\varphi(u) = v$.
I.e., a graph $\Gamma = (V, E)$ is vertex-transitive if and only if $Aut(\Gamma)$ has exactly one orbit on $V(\Gamma)$.

Examples of vertex-transitive graphs:

- ⋆ a vertex-transitive graph of order n is necessarily k-regular for some $0 \leq k \leq n-1$
- ⋆ complete graphs K_n, $n \geq 2$
- ⋆ the Petersen graph
- ⋆ **Johnson graphs** $J(n, k)$
- ⋆ **Kneser graphs** $K(n, k)$

Vertex-Transitive Graphs

Problem: Show that the Johnson and Kneser graphs are vertex-transitive.

- ⋆ Is a strongly regular graph necessarily vertex-transitive?
- ⋆ Is a Latin square graph necessarily vertex-transitive?
- ⋆ Is a Latin square graph whose Latin square is a multiplication table for a group necessarily vertex-transitive?
- ⋆ Is the group of automorphisms of a group G necessarily transitive on the elements of G?

Transitive permutation groups

Definition

A group $G \leq \mathbb{S}_X$ is **transitive (acts transitively) on** X, if for every pair of vertices $x, y \in X$ there exists a permutation φ, such that $\varphi(x) = y$.

Examples:

- $\star$ the full symmetric group $\mathbb{S}_X$ acts transitively on X; what is the order of $Stab_{\mathbb{S}_X}(x)$? how many permutations map x to y?
- $\star$ the alternating group $\mathbb{A}_X$ acts transitively on X; what is the order of $Stab_{\mathbb{A}_X}(x)$? how many permutations map x to y?

More examples of transitive permutation groups

- ⋆ if G is a group, let $G_L = \{\sigma_g \mid g \in G\}$, where $\sigma_g(h) = gh$, for all $h \in G$
- ⋆ $G_L \leq \mathbb{S}_G$ and G_L acts transitively on G; what is the order of $Stab_{G_L}(h)$? how many permutations σ_g map h to h'?
- ▶ this is called the **(left) regular representation of** G

More examples of transitive permutation groups

- ⋆ Let G be a group, H be a subgroup of G, and let X be the set of left cosets of H in G

$$X = \{1_G H = g_1 H, g_2 H, g_n H\},$$

where $n = [G : H]$

- ⋆ for every element $g \in G$ define a permutation $\sigma_{g,H}$ of the set X by the rule $\sigma_{g,H}(g_i H) = g g_i H$; is it a permutation?
- ⋆ then $\{\sigma_{g,H} \mid g \in G\} \leq \mathbb{S}_X$ and $\{\sigma_{g,H} \mid g \in G\}$ acts transitively on X
- ⋆ it is important to realize that the order of $\{\sigma_{g,H} \mid g \in G\}$ may be smaller than $|G|$, since the homomorphism $\Phi_H : g \to \sigma_{g,H}$ may not be injective; is it a homomorphism?
- ⋆ Prove that Φ_H is injective if and only if H is **core-free** in G, i.e.,

$$\bigcap_{g \in G} gHg^{-1} = \langle 1_G \rangle$$

This is definitely enough for today!