Ako vlastné čísla utláčajú Mooreovské grafy

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Eigenvalues and eigenvectors

F-field, $A \in M_{n\times n}(F)$, $\lambda \in \overline{F}$, $\alpha \in F^n \setminus \{0\}$.

Characteristic polynomial of A is the polynomial $|A - xI|$.

If $A\alpha = \lambda\alpha$, then λ is an *eigenvalue* of A and α is an *eigenvector* of A corresponding to λ.

Lemma. *λ is an eigenvalue of A iff $|A - \lambda I| = 0$.*

Lemma. *Let $A^T = A \in M_{n\times n}(\mathbb{R})$. Then every eigenvalue of A is real and $\mathbb{R}^n$ has an orthogonal basis consisting of eigenvectors of A.*

Moore bound

Let Γ be a graph with maximum degree k and diameter d. Then

$$|\Gamma| \leq 1 + k + k(k-1) + k(k-1)^2 + \cdots + k(k-1)^{d-1}.$$

If the Moore bound is attained, then Γ is a *Moore graph*.

Moore graph is always regular.

Trivial Moore graphs

- $d = 1$: $\Gamma = K_{k+1}$
- $k = 2$: $\Gamma = C_{2d+1}$

Nontrivial Moore graphs

- A. J. Hoffman and R.R. Singleton (1960)

- E. Bannai and T. Ito (1973)

- R.M. Damerell (1973)

Theorem. *Moore graph with* $d = 2$ *exists only if* k *is equal to* 2, 3, 7 *or (maybe)* 57.

Theorem. *Nontrivial Moore graphs exist only for* $d = 2$.

Alternative characterizations of Moore graphs

Lemma. *A k-regular graph with diameter 2 is Moore iff*

$$a \sim b \;\Rightarrow\; |N(a) \cap N(b)| = 0, \tag{1}$$

$$a \not\sim b \;\Rightarrow\; |N(a) \cap N(b)| = 1. \tag{2}$$

Lemma. *Let A be the adjacency matrix of k-regular graph Γ with diameter 2. Then Γ is Moore iff*

$$A^2 + A - (k-1)I = J, \tag{3}$$

where J is the all 1 matrix.

Eigenvalues of Moore graphs

Lemma. *Let λ be an eigenvalue of a k-regular Moore graph Γ. Then $\lambda \in \{k, \frac{-1 \pm \sqrt{4k-3}}{2}\}$.*

Proof . $(1, 1, \ldots, 1)^T$ is an eigenvector of Γ corresponding to k.

Let α be an eigenvector of of Γ corresponding to λ, orthogonal to $(1, 1, \ldots, 1)^T$. Then by (3) we have $(\lambda^2 + \lambda - (k-1))\alpha = 0\alpha\Box$

The eigenvalues $\frac{-1+\sqrt{4k-3}}{2}$ and $\frac{-1-\sqrt{4k-3}}{2}$ of Γ are usually denoted by r and s, respectively.

Multiplicities of eigenvalues

Lemma. *Let* Γ *be a Moore graph. Then the multiplicities of eigenvalues* k, r *and* s *are* 1, f *and* g, *where*

$$f, g = \frac{1}{2}\left(k^2 \pm \frac{k^2 - 2k}{\sqrt{4k-3}}\right).$$

Proof . Multiplicity of k is one. We know that $1+f+g = |\Gamma| = k^2+1$ and $1k+fr+gs = Tr(A) = 0$. By solving these equations we obtain $g = (rk^2+k)/(r-s)$ and $f = k^2 - g$. Substituting $r = \frac{-1+\sqrt{4k-3}}{2}$ and $r - s = \sqrt{4k-3}$ gives the desired result$\square$

Integral condition

Lemma. *Let Γ be a Moore graph. Then $\frac{k^2-2k}{\sqrt{4k-3}}$ is an integer congruent to k mod 2.*

Case I: $k = 2$ and $\Gamma = C_5$.

Case II: $k^2 - 2k \neq 0$. Then $\sqrt{4k-3}$ is an odd integer and both r and s are integers. We have $r + s = -1$, $k - 1 = r(1 + r)$ and $\sqrt{4k-3} = r - s = 2r + 1$. Therefore

$$2r + 1 | (r^2 + r + 1)(r^2 + r - 1). \qquad (4)$$

Case II (cont.)

As $2r+1$ is odd, we can multiply the right-hand side of (4) by 16 to obtain

$$\begin{aligned} 2r+1 &\mid (4r^2+4r+4)(4r^2+4r-4) \\ 2r+1 &\mid ((2r+1)^2+3)((2r+1)^2-5) \\ 2r+1 &\mid 3\cdot 5 \end{aligned}$$

Solution $r=0$ gives $k=1$ and $\Gamma = K_2$ with diameter 1.

Solutions $r=1$, 2 and 7 give $k=3$, 7 and 57, respectively.

Moore graphs with diameter 2

- $k = 2$, $v = 5$: a pentagon

- $k = 3$, $v = 10$: the Petersen graph

- $k = 7$, $v = 50$: the Hoffman-Singleton graph

- $k = 57$, $v = 3250$: ???

The missing Moore graph

From now on, let Γ denote a Moore graph of degree 57. If it exists, it has 3250 vertices and its eigenvalues are 57, 7 and -8 with multiplicities 1, 1729 and 1520, respectively.

Fixed points of automorphisms

Lemma. M. Aschbacher (1971) *Let* X *be an automorphism group of* Γ. *Then* $\mathsf{Fix}(X)$ *satisfies* (1) *and* (2). *In particular,* $\mathsf{Fix}(X)$ *is one of the following:* $\emptyset$, *an isolated vertex, a pentagon, the Petersen graph, the Hoffman-Singleton graph or a star* $K_{1,n}$.

Lemma. M. Aschbacher (1971) *Let* x *be an involutory automorphism of* Γ. *Then* $|\mathsf{Fix}(x)| \in \{56, 58\}$.

Lemma. D. Higman (??) *Let* x *be an involutory automorphism of* Γ. *Then* $|\mathsf{Fix}(x)| = 56$.

Fixed points of automorphisms

Proposition. *Let X be an automorphism group of Γ of order p^n.*
1) If $\mathrm{Fix}(X) = \emptyset$, *then* $p \in \{5, 13\}$.
2) If $\mathrm{Fix}(X) = \{a\}$, *then* $p \in \{3, 19\}$.
3) If $\mathrm{Fix}(X)$ *is a star, then* $p \in \{2, 7\}$.
4) If $\mathrm{Fix}(X)$ *is a pentagon, then* $p \in \{5, 11\}$.
5) If $\mathrm{Fix}(X)$ *is the Petersen graph , then* $p = 3$.
6) If $\mathrm{Fix}(X)$ *is the Hoffman-Singleton graph, then* $p = 5$.

Corollary. $|\mathrm{Aut}(\Gamma)| < 2 \cdot 3^4 \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19 = 33699206250$.

Theorem. A.A. Makhnev and D.V. Paduchikh (2001) *If* $\mathrm{Aut}(\Gamma)$ *is even, then* $|\mathrm{Aut}(\Gamma)| \leq 550$.

Adjacency matrix of an automorphism group

Lemma. *Let X be an automorphism group of Γ having orbits $O_1, O_2, \ldots, O_m$ of size $s_1, s_2, \ldots, s_m$, respectively. Let $a \in O_i$. Then the number $b_{i,j} = |N(a) \cap O_j|$ does not depend on a and the matrix $B = \|b_{i,j}\|$ satisfies:*
1) $s_i b_{i,j} = s_j b_{j,i}$;
2) $B^2 + B - 56I = (1, 1, \ldots, 1)^T (s_1, s_2, \ldots, s_m)$,
3) eigenvalues of B belong to $\{57, 7, -8\}$.

Corollary.

$$Tr(B) \equiv 80 - 8m \mod 15. \tag{5}$$

Applications

Lemma. *Let* x *be an automorphism of* Γ *of order* 2 *or* 7*. Then* $|\,\mathsf{Fix}(x)| < 58$*.*

Lemma. *Let* x *be an automorphism of* Γ *of order* 3*. Then* $|\,\mathsf{Fix}(x)| = 10$*.*

Lemma. *Let* X *be an automorphism group of* Γ *with* $\mathsf{Fix}(X)$ *equal to the Hoffman-Singleton graph. Then* $|X| = 5$*.*

New bounds

Theorem. M.M. and J. Širáň
Let G be the automorphism group of Γ of odd order.
Then $|G| \in \{1, 3, 5, 7, 11, 13, 15, 19, 21, 25, 27, 35, 39, 45, 55, 57, 75, 81, 125, 135, 147, 171, 275, 375\}$.

Theorem. M.M. and J. Širáň
Let G be the automorphism group of Γ of even order.
Then $|G| \in \{2, 6, 10, 14, 18, 22, 38, 50, 54, 110\}$.

Thank You