Classification problems in the topology of high-dimensional manifolds

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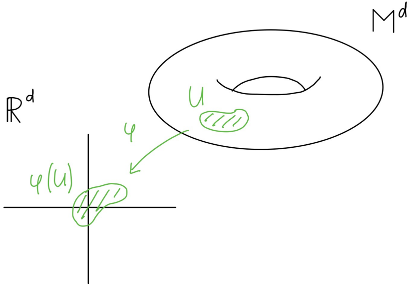
Manifolds

Definition

A topological manifold M of dimension $d \geq 0$ is
a second countable hausdorff locally euclidean space of dimension d.



The structure set

Definition

$\mathcal{S}^{\mathrm{TOP},s}(M)$ = the simple structure set of a manifold M:

- (N, f), $f \colon N \xrightarrow{\simeq_s} M$
- $(N, f) \sim (N', f')$ if exists $h \colon N \xrightarrow{\cong} N'$ s.t. $f' \circ h \simeq f$.

Definition

$\mathcal{S}_\partial^{\mathrm{TOP},s}(M \times D^k)$ = the k-th higher simple structure set of a manifold M:

- (N, f), $f \colon (N, \partial N) \xrightarrow{(\simeq_s, \cong)} (M \times D^k, M \times \partial D^k)$
- $(N, f) \sim (N', f')$ if exists $h \colon N \xrightarrow{\cong} N'$ s.t. $f' \circ h \simeq f$ rel ∂N.

Classification problems

Definition (automorphism spaces)

$$\widetilde{\mathcal{S}}^{\mathrm{TOP},s}(M)_k = \{h \colon N \xrightarrow{\simeq_s} M \times \Delta^k \mid h(N(\sigma)) \subseteq M \times \sigma,\ \forall \sigma \in \Delta^k\},$$

$$\mathrm{TOP}(M)_k = \{h \colon M \times \Delta^k \xrightarrow{\cong} M \times \Delta^k \mid h \text{ is over } \Delta^k\},$$

$$\mathrm{G}^s(M)_k = \{h \colon M \times \Delta^k \xrightarrow{\simeq_s} M \times \Delta^k \mid h \text{ is over } \Delta^k\},$$

$$\widetilde{\mathrm{TOP}}(M)_k = \{h \colon M \times \Delta^k \xrightarrow{\cong} M \times \Delta^k \mid h(M \times \sigma) \subseteq M \times \sigma,\ \forall \sigma \in \Delta^k\},$$

$$\widetilde{\mathrm{G}}^s(M)_k = \{h \colon M \times \Delta^k \xrightarrow{\simeq_s} M \times \Delta^k \mid h(M \times \sigma) \subseteq M \times \sigma,\ \forall \sigma \in \Delta^k\}.$$

Relations

$$\begin{tikzcd}
 & \mathrm{TOP}(M) \arrow[r, "\text{alg K-thy}"] \arrow[d] & \widetilde{\mathrm{TOP}}(M) \arrow[d, "\text{surgery}"] \\
\text{homotopy} \arrow[r, dotted] & \mathrm{G}^s(M) \arrow[r, "\simeq"'] & \widetilde{\mathrm{G}}^s(M)
\end{tikzcd}$$

The surgery exact sequence

Theorem (Browder-Novikov-Sullivan-Wall $\sim$ [1966])

For an n-manifold M with $n \geq 5$ we have

$$\cdots \to \mathcal{N}_{\partial}^{\mathrm{TOP}}(M \times I) \xrightarrow{\theta} L_{n+1}^{s}(\mathbb{Z}G) \xrightarrow{\partial} \mathcal{S}^{\mathrm{TOP},s}(M) \xrightarrow{\eta} \mathcal{N}^{\mathrm{TOP}}(M) \xrightarrow{\theta} L_{n}^{s}(\mathbb{Z}G),$$

where $G = \pi_1(M)$.

Explanation

- $\mathcal{N}^{\mathrm{TOP}}(M)$ - normal cobordism - gen. cohomology theory
- $L_n^s(\mathbb{Z}G)$ - Witt group of quadratic forms
- θ - the surgery obstruction map

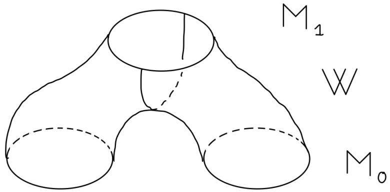
Cobordism

Definition

An (oriented) cobordism $(W; M_0, M_1)$ between M_0 and M_1 is a compact $(d+1)$-dim (oriented) manifold W such that

$$\partial W \cong M_0^- \coprod M_1.$$

Cobordism:



Surgery

Definition

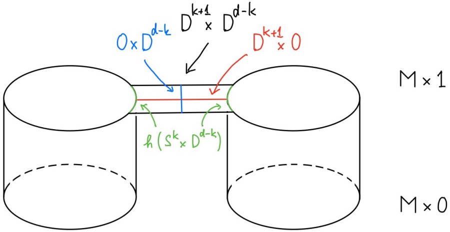
Let M be a closed d-dim manifold and let $h\colon S^k \times D^{d-k} \hookrightarrow M$.
The manifold M' is obtained from M by surgery along h:

$$M' := (M \smallsetminus h(S^k \times D^{d-k})) \cup_{\partial h} D^{k+1} \times S^{d-k-1}.$$

The trace of the surgery on M along h is the $(d+1)$-dim mfd

$$W := M \times [0,1] \cup_h D^{k+1} \times D^{d-k} \quad \text{with} \quad \partial W = M \coprod M'.$$

Surgery:



Lens spaces

Definition

$$L_N(k_1, \ldots, k_d) := S^{2d-1}/\sim = S(\mathbb{C}^d)/\sim$$
$$(z_1, \ldots, z_d) \sim (z_1 \cdot e^{2\pi i k_1/N}, \ldots, z_d \cdot e^{2\pi i k_d/N}).$$

Theorem [A1] 1.2

Let L_N^{2d-1} be a lens space with $\pi_1(L_N^{2d-1}) \cong \mathbb{Z}/N$ where $N = 2^K$, $d \geq 3$. Then for $c = \lfloor (d-1)/2 \rfloor$ we have

$$\mathcal{S}^s(L_N^{2d-1}) \cong \bar{\Sigma} \oplus \bar{T} \cong \bar{\Sigma} \oplus \bigoplus_{i=1}^{c} \mathbb{Z}/2 \oplus \bigoplus_{i=1}^{c} \mathbb{Z}/2^{\min\{K,2i\}}$$

where $\bar{\Sigma}$ is a free abelian group of rank $\begin{cases} N/2 - 1 & d = 2e+1 \\ N/2 & d = 2e \end{cases}$.

Surgery exact sequence for lens spaces

Known results were

$$0 \to \widetilde{L}^s_{2d}(\mathbb{Z}G) \xrightarrow{\partial} \mathcal{S}^s(L^{2d-1}_N) \xrightarrow{\eta} \widetilde{\mathcal{N}}(L^{2d-1}_N) \to 0$$

$$\widetilde{L}^s_n(\mathbb{Z}G) \cong \begin{cases} 4 \cdot R^+_{\mathbb{C}}(G)/\langle reg \rangle & n \equiv 0 \pmod 4 \text{ (G-sign, real)} \\ 0 & n \equiv 1 \pmod 4 \\ 4 \cdot R^-_{\mathbb{C}}(G) & n \equiv 2 \pmod 4 \text{ (G-sign, imaginary)} \\ 0 \text{ or } \mathbb{Z}/2 & n \equiv 3 \pmod 4 \text{ (codimension 1 Arf)} \end{cases}$$

$$\widetilde{\mathcal{N}}(L^{2d-1}_N) \cong \bigoplus_{i=1}^{c} \mathbb{Z}/N \oplus \bigoplus_{i=1}^{c} \mathbb{Z}/2, \quad \textit{where } c = \lfloor (d-1)/2 \rfloor.$$

The ρ-invariant

Definition [Atiyah-Singer-III(1968)]

Let N be a closed topological $(2d-1)$-dim manifold with a map $\lambda(N)\colon N \to BG$ where G is a finite group. Define

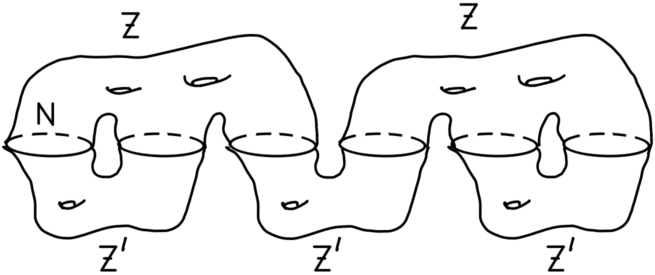
$$\rho(N, \lambda(N)) := \frac{1}{r} \cdot \text{G-sign}(\widetilde{Z}) \in \mathbb{Q}R^{(-1)^d}(G)/\langle \text{reg} \rangle =: \mathbb{Q}R^{(-1)^d}_{\widehat{G}}$$

for some $r \in \mathbb{N}$ and $(Z, \partial Z)$ such that $\partial Z = r \cdot N$ and there is $\lambda(Z)\colon Z \to BG$ restricting to $r \cdot \lambda(N)$ on ∂Z.

Useful formulas

$$f := \frac{1+\chi}{1-\chi}, \quad (1-\chi)^{-1} = -\frac{1}{N}(1+2\cdot\chi+3\cdot\chi^2+\cdots+N\cdot\chi^{(N-1)}) \in \mathbb{Q}R_{\widehat{G}}.$$

The ρ-invariant is well defined



Proof

Key diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{L}^s_{2d}(\mathbb{Z}G) \arrow[r, "\partial"] \arrow[d, "\cong"', "\text{G-sign}"] & \mathcal{S}^s(L^{2d-1}_N) \arrow[r, "\eta"] \arrow[d, "\widetilde{\rho}"] & \widetilde{\mathcal{N}}(L^{2d-1}_N) \arrow[r] \arrow[d, "{[\widetilde{\rho}]}"] & 0 \\
0 \arrow[r] & 4 \cdot R^{(-1)^d}_{\widehat{G}} \arrow[r] & \mathbb{Q}R^{(-1)^d}_{\widehat{G}} \arrow[r] & \mathbb{Q}R^{(-1)^d}_{\widehat{G}} / 4 \cdot R^{(-1)^d}_{\widehat{G}} \arrow[r] & 0
\end{tikzcd}$$

Observation

$$\mathcal{S}^s(L^{2d-1}_N) \cong \mathrm{im}\ \widetilde{\rho} \oplus \ker([\widetilde{\rho}])$$

Key formula

$$d = 2e \ : \ [\widetilde{\rho}](t) = \sum_{i=1}^{e-1} 8 \cdot t_{4i} \cdot f^{d-2i-2} \cdot (f^2 - 1)$$

N odd and the higher structure sets

Theorem (Wall [1970]), cf Theorem [A3] 6.1

Let L_N^{2d-1} be a lens space with $\pi_1(L_N^{2d-1}) \cong \mathbb{Z}/N$ where N is odd, $d \geq 3$. Then

$$\mathcal{S}^s(L_N^{2d-1}) \cong \bar{\Sigma}$$

where $\bar{\Sigma}$ is a free abelian group of rank $(N-1)/2$.

Theorem [A4] 1.2

Let L_N^{2d-1} be a lens space with $\pi_1(L_N^{2d-1}) \cong \mathbb{Z}/N$, $N = 2^K$, $d = 2e \geq 3$. Then for $c = \lfloor (d-1)/2 \rfloor$ we have

$$\mathcal{S}^s_\partial(L_N^{2d-1} \times D^{4l}) \cong \bar{\Sigma} \oplus \mathbb{Z} \oplus \bar{T} \cong \bar{\Sigma} \oplus \mathbb{Z} \oplus \bigoplus_{i=1}^{c} \mathbb{Z}/2 \oplus \bigoplus_{i=1}^{c} \mathbb{Z}/2^{\min\{K,2i\}}$$

where $\bar{\Sigma}$ is a free abelian group of rank $N/2$.

Algebraic surgery exact sequence (Ranicki)

Symmetric structure on a chain complex C

$$n-\text{cycle: } \varphi \in W^{\%}(C) := \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C).$$

Key ideas

Cobordism groups of quadratic chain cplxs in cats with chain duality.

GSES $\rightsquigarrow$ ASES

$$\begin{tikzcd}
L_{n+1}(\mathbb{Z}[\pi]) \arrow[r, "\partial"] \arrow[d, "="] & \mathcal{S}^{\mathrm{TOP}}(X) \arrow[r] \arrow[d, "\mathbf{qsign}_X", "\cong"'] & \mathcal{N}^{\mathrm{TOP}}(X) \arrow[r, "\mathbf{qsign}"] \arrow[d, "\mathbf{qsign}_X", "\cong"'] & L_n(\mathbb{Z}[\pi]) \arrow[d, "="] \\
L_{n+1}(\mathbb{Z}[\pi]) \arrow[r, "\partial"] & \mathbb{S}_{n+1}(X) \arrow[r] & H_n(X; \mathbf{L}_\bullet\langle 1\rangle) \arrow[r, "\mathrm{asmb}"] & L_n(\mathbb{Z}[\pi])
\end{tikzcd}$$

The additivity of the ρ-invariant

Theorem [B1] 1.1

Let M be a closed oriented topological manifold of dim $2d-1 \geq 5$ with a map $\lambda(M)\colon M \to BG$ where G is a finite group. Then

$$\widetilde{\rho}\colon \mathcal{S}(M) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d} \quad \text{given by} \quad h\colon N \to M \ \mapsto \rho(N) - \rho(M)$$

is a homomorphism of abelian groups.

Theorem [B1] 1.3

Let M be a closed oriented topological manifold of dim n with a map $\lambda(M)\colon M \to BG$ for a finite group G, and let $n + l = 2d - 1 \geq 5$. Then

$$\widetilde{\rho}_\partial\colon \mathcal{S}_\partial(M \times D^l) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d}$$

is a homomorphism of abelian groups.

Proof

Theorem [B1] 1.5

Let M be a closed topological manifold of dimension $(2d-1) \geq 5$ with a reference map $\lambda\colon M \to BG$ for a finite group G.
Then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{S}(M) \arrow[rr, "CW^2"] \arrow[dr, "\widetilde{\rho}"'] & & \mathcal{S}_\partial(M \times D^8) \arrow[dl, "\widetilde{\rho}_\partial"] \\
& \mathbb{Q}R^{(-1)^d}_{\widehat{G}} &
\end{tikzcd}$$

Manifold sets

Definition/Proposition

$$\mathcal{M}^s(M) := \{N \mid N \simeq_s M\}/\cong \xrightarrow{\equiv} \mathcal{S}^s(M)/\mathrm{G}^s(M)$$

where

$$\mathcal{S}^s(M) \times \mathrm{G}^s(M) \to \mathcal{S}^s(M), \quad ([f : N \to M], [g]) \mapsto [g \circ f : N \to M].$$

Theorem [B4] 1.4

If $\pi_1(M) = \{e\}$ and $n \geq 5$ then

$$|\mathcal{M}(M)| = \infty \ \Leftrightarrow |\mathcal{S}(M)| = \infty.$$

In such a case for some $0 < 4k < n$ the set $\mathrm{div}_k(\mathcal{M}(M))$ is infinite.

h-cobordism and s-cobordism

h-Cobordism Theorem [Smale(1961)]

Every h-cobordism over a 1-ctd manifold M_0 with $\dim(M_0) \geq 5$ is trivial.

$$K_1(\mathbb{Z}G) := \mathrm{GL}(\mathbb{Z}G)/[\mathrm{GL}(\mathbb{Z}G), \mathrm{GL}(\mathbb{Z}G)].$$

The Whitehead group of G is defined as

$$\mathrm{Wh}(G) := \mathrm{coker}\,(G \times \{\pm 1\} \to K_1(\mathbb{Z}G)).$$

s-Cobordism Theorem [Milnor(1966)]

Let M_0 be a manifold of dim $n \geq 5$ with $\pi = \pi_1(M_0)$.
Then the Whitehead torsion defines a bijection

$$\tau \colon \{h - \mathrm{cobs}\ (W; M_0, M_1) \text{ over } M_0\} \Big/ \cong \quad \xrightarrow{\cong} \quad \mathrm{Wh}(\pi).$$

Algebraic K-theory

Definitions (Quillen, Waldhausen [1970-1980])

$$\mathbf{K}(R) := K_0(R) \times \mathrm{BGL}(R)^+ \quad \rightsquigarrow \quad R - \text{ring}$$

$$\mathbf{K}(\mathbb{A}) := K_0(\mathbb{A}) \times \mathrm{BGL}(\mathbb{A})^+ \quad \rightsquigarrow \quad \mathbb{A} - \text{ring spectrum}$$

$$A(M) := \mathbf{K}(\mathbb{S}[\Omega M]) \quad \rightsquigarrow \quad \pi_0 \mathbb{S}[\Omega M] = \mathbb{Z}[\pi_1 M]$$

$$M_+ \wedge A(*) \to A(M) \to \mathbf{Wh}^{\mathrm{TOP}}(M)$$

Theorem (SPHCT) [Waldhausen-Jahren-Rognes (2013)]

$$\mathcal{H}^{\mathrm{TOP}}(M) \simeq \Omega \mathbf{Wh}^{\mathrm{TOP}}(M).$$

Theorem [Weiss-Williams (1988)]

$$\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \xrightarrow{\sim (n/3)-\mathrm{ctd}} \Omega^\infty [\mathcal{H}^{\mathrm{TOP}}(M)_{h\mathbb{Z}/2}]$$

Algebraic K-theory of $X \times S^1$

Theorem [Bass-Heller-Swan (1964)]

$$\mathbf{K}(\mathbb{Z}[\pi \times \mathbb{Z}]) \simeq \mathbf{K}(\mathbb{Z}[\pi]) \times B\mathbf{K}(\mathbb{Z}[\pi]) \times \mathrm{N}\mathbf{K}(\mathbb{Z}[\pi]) \times \mathrm{N}\mathbf{K}(\mathbb{Z}[\pi])$$

$NK(R)$ is defined using nilpotent matrices over R

Theorem [HKVWW (2001)]

$$A(X \times S^1) \simeq A(X) \times BA(X) \times \mathrm{N}A(X) \times \mathrm{N}A(X)$$

Theorem [Farrell (1977)]

$\pi_* \, \mathrm{N}\mathbf{K}(R)$ are either trivial or not finitely generated as abelian groups.

$$\iota_n \colon R[t^n] \hookrightarrow R[t] \begin{cases} \rightsquigarrow & (\iota_n)^* \colon \mathbf{K}(R[t]) \to \mathbf{K}(R[t^n]) \\ \rightsquigarrow & (\iota_n)_* \colon \mathbf{K}(R[t^n]) \to \mathbf{K}(R[t]) \end{cases}$$

Nil terms in A-theory

Theorem [C1] 1.2

For p an odd prime there is a $(4p-7)$-connected map

$$\bigvee_{\pm n \in \mathbb{N}_\times} \Sigma^{2p-2} \mathbb{HF}_p \wedge (S^1(\pm n)_+) \longrightarrow \mathrm{NA}_\pm(*)^\wedge_p.$$

Theorem [C1] 1.3

The $\mathbb{Z}_p[\mathbb{N}_\times]$-module $\pi_{2p-2}\,\mathrm{NA}_\pm(*)^\wedge_p$ is (finitely) generated by $\beta_{\pm 1}$, and the $\mathbb{Z}_p[\mathbb{N}_\times]$-module $\pi_{2p-1}\,\mathrm{NA}_\pm(*)^\wedge_p$ is not finitely generated.

Application

For M neg curved, $\dim = n \geq 10$, $1 \leq j < \varphi(n)$, with $\varphi(n) \sim n/3$, we have

$$\pi_j \mathrm{TOP}(M) \cong \bigoplus_T \pi_{j+2}\,\mathrm{NA}(*)$$