Complex analysis II. – Homework 1

Submission date: September 30, 2014

(AF) will denote a reference to the book by M. Ablowitz and A. Fokas; usually by a problem number (chapter.section.problem), or by page(s) where you can read more about given topic.

1. (AF p. 34 - 35) Derive the Cauchy-Riemann equations for polar coordinates r and θ:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

2. Show that for an analytic function F we have

$$\int_C F'(z)dz = F(b) - F(a),$$

where a and b are initial and terminal points of the contour C.

Compare with

$$\int_C \nabla f \cdot d\vec{s} = f(b) - f(a),$$

for the contour integral of the gradient of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$. Such function f is called a *potential* of the vector filed ∇f.

3. (AF 2.5.4, 2.6.1) Find values of integrals

$$a) \quad \oint_C \frac{e^{z^2}}{z^2}dz, \qquad b) \quad \oint_C \frac{e^z}{z}dz,$$

where C is a simple closed curve enclosing the origin. Use Taylor series as necessary.

4. (AF 2.5.5) We wish to evaluate the integral $I = \int_0^\infty e^{ix^2}dx$. Consider the countour $I_R = \oint_{C_{(R)}} e^{iz^2}dz$, where $C_{(R)}$ is the closed circular sector in the upper half plane with boundary points 0, R and $Re^{i\pi/4}$. Show that $I_R = 0$ and that $\lim_{R\to\infty} \int_{C_{1(R)}} e^{iz^2}dz = 0$, where $C_{1(R)}$ is the line integral along the circular sector from R to $Re^{i\pi/4}$. Using the well-known result, $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$, deduce that $I = e^{i\pi/4}\sqrt{\pi}/2$.

5. (AF 2.6.3) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{(x+i)^2}dx$$

by considering $\oint_{C_{(R)}} (1/(z+i)^2)dz$, where $C_{(R)}$ is the closed semicircle in the upper half plane with corners $z = -R$ and $z = R$, plus the x axis.

What would change if we integrated along the semicircle in the lower half plane? Try to compare your result with the fact that $(-1/(z+i))' = 1/(z+i)^2$.

6. Suppose that functions f and g are analytic inside and on the boundary of the unit circle. Suppose further that they attain same values on the boundary, i.e. $f(z) = g(z)$ for $|z| = 1$. Show that $f \equiv g$ inside the unit disk.