Complex Analysis II. – Homework 2

Submission date: October 7, 2014

---

1. Let C be the unit circle and D its interior, i.e. the disk with radius 1. Investigate what happens when we apply the Cauchy Integral Formula to the function $1/z$, which has a pole at zero.

a) Find

$$f(z) = \oint_C \frac{1}{\zeta(\zeta - z)} d\zeta, \qquad \text{for} \quad |z| \neq 1.$$

Hint: Compute the residues of the function $1/\zeta(\zeta - z)$ at points 0 and z or use the decomposition $1/\zeta(\zeta - z) = \frac{1}{z}(1/(\zeta - z) - 1/\zeta)$. Treat the cases $|z| < 1$ and $|z| > 1$ separately.

b) For ζ lying on the unit circle we have $1/\zeta = \bar{\zeta}$. Use the generalized Cauchy integral formula for the function $g(z) = \bar{z}$. Deduce the following identity:

$$-\pi \bar{z} = \int\int_D \frac{dS}{\zeta - z}, \qquad \text{for} \quad |z| < 1.$$

2. (AF 2.6.8) Find the $\bar{\partial}$ (dbar) derivative of the function $z\bar{z}$ (that is r^2). Verify the generalizeed Cauchy formula inside a circle of radius R by reducing to the identity:

$$-\pi \bar{z} = \int\int_{D_R} \frac{dS}{\zeta - z} = \int\int_{D_R} \frac{d\xi d\eta}{\zeta - z} \equiv I,$$

where D_R is a disk of radius R (see problem 1; note that the value of the integral does not depend on the radius R). Verify by a direct calculation that such equality indeed holds.

Hint: Transform the integral I to polar coordinates $\zeta = \xi + i\eta = re^{i\theta}$, and find

$$I = \int_0^{2\pi} \int_0^R \frac{r\, dr\, d\theta}{re^{i\theta} - z}.$$

In the θ integral, change the variables to $u = e^{i\theta}$, and use $du = ie^{i\theta} d\theta$, leading to $\int_0^{2\pi} f(\theta) d\theta = \frac{1}{i} \oint_{C_1} f(u) \frac{du}{u}$, where C_1 is the unit circle. The residue calculus can be used to evaluate such integrals, or we can integrate along small circles around the poles ($u = 0$ and $u = z/r$, compare with problem 1, or section 2.5 in AF)

Show that we have

$$I = 2\pi \int_0^R r\, dr \left[-\frac{1}{z} + \frac{1}{z} H\left(1 - \frac{|z|}{r}\right) \right],$$

where $H(x) = \{1 \text{ if } x > 0,\ 0 \text{ if } x < 0\}$ (this corresponds to the condition $z \in D_r$). Then show that $I = -\pi |z|^2 / z = -\pi \bar{z}$ as is required.

3. (AF 2.6.10) In Cauchy's Integral Formula, take the contour to be a circle of unit radius centered at the origin. Let $\zeta = e^{i\theta}$ to deduce

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - z} d\theta,$$

where z lies inside the circle. Explain why we have

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - 1/\bar{z}} d\theta$$

and use $\zeta = 1/\bar{\zeta}$ to show

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \left(\frac{\zeta}{\zeta - z} \pm \frac{\bar{z}}{\bar{\zeta} - \bar{z}} \right) d\theta$$

whereupon, using the plus sign

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \frac{(1 - |z|^2)}{|\zeta - z|^2} d\theta.$$

(a) Deduce the "Poisson formula" for the real part of $f(z)$, $\mathrm{Re}\, f = u(r, \phi)$, where $z = re^{i\phi}$

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \frac{1 - r^2}{[1 - 2r\cos(\phi - \theta) + r^2]} d\theta,$$

where $u(\theta) = u(1, \theta)$.

(b) If we use the minus sign in the formula for $f(z)$ above, show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \left(\frac{1 + r^2 - 2re^{i(\theta - \phi)}}{1 - 2r\cos(\phi - \theta) + r^2} \right) d\theta$$

and by taking the imaginary part

$$v(r, \phi) = C + \frac{1}{\pi} \int_0^{2\pi} u(\theta) \frac{r \sin(\phi - \theta)}{[1 - 2r\cos(\phi - \theta) + r^2]} d\theta,$$

where $C = \frac{1}{2\pi} \int_0^{2\pi} v(1, \theta) d\theta = v(r = 0)$.

(c) Show that

$$\begin{aligned} \frac{2r\sin(\phi - \theta)}{1 - 2r\cos(\phi - \theta) + r^2} &= \mathrm{Im} \left(\frac{1 - r^2 + 2ir\sin(\phi - \theta)}{1 + r^2 - 2r\cos(\phi - \theta)} \right) \\ &= \mathrm{Im} \left(\frac{\zeta + z}{\zeta - z} \right), \end{aligned}$$

and therefore the result for $u(r, \phi)$ and $v(r, \phi)$ from parts (a) and (b) may be expressed as

$$\begin{aligned} u(r, \phi) &= \frac{\mathrm{Re}}{2\pi} \int_0^{2\pi} u(\theta) \frac{\zeta + z}{\zeta - z} d\theta, \\ v(r, \phi) &= v(0) + \frac{\mathrm{Im}}{2\pi} \int_0^{2\pi} u(\theta) \frac{\zeta + z}{\zeta - z} d\theta. \end{aligned}$$

This example illustrates that prescribing the real part of (an analytic) $f(z)$ on $|z| = 1$ determines uniquely (a) the real part of $f(z)$ everywhere inside the circle and (b) the imaginary part of $f(z)$ inside circle to within a constant. We *cannot* arbitrarily specify both the real and imaginary parts of an analytic function on $|z| = 1$.

Try (without much of a computation) to determine which analytic function corresponds to $u(1, \theta) = \mathrm{Re}\,(1/z)$ from the Problem 1.