Complex Analysis II. – Homework 3

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1. (AF 4.3.1) a) Use principal value integrals to show that

$$\int_0^\infty \frac{\cos kx - \cos mx}{x^2} dx = \frac{-\pi}{2}(|k| - |m|), \qquad k, m \,\text{real}.$$

b) Let $k = 2$ and $m = 0$ to deduce that

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

Could this integral be evaluated by some other method?

2. (following AF 4.3.5) Consider a function $F(z)$

$$F(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where C is a contour, typically infinite (e.g. the real axis) or closed (e.g. a circle). Then the "plus" and "minus" projections of $F(z)$ at $z = \zeta_0$ are defined by the following limit:

$$F^{\pm}(\zeta_0) = \lim_{z \to \zeta_0^{\pm}} \left[\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta \right],$$

where ζ_0 lies on C, $\lim_{z \to \zeta_0^+}$ denotes the limit from points z inside (+) or left (+) of the contour C, similarly for the "minus" limit from outside (–) or right (–).

By a direct computation find $F^{\pm}(\zeta_0)$ for $f(\zeta) = 1/(\zeta^2 + 1)$ and C being the real axis $(-\infty, \infty)$. Use the theory from the lecture as necessary.

3. (following AF 4.3.15) In the last homework we deduced the formula

$$v(r, \phi) = v(r = 0) + \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \frac{2r \sin(\phi - \theta)}{[1 - 2r \cos(\phi - \theta) + r^2]} d\theta,$$

where $u(\theta)$ is given on the unit circle and the harmonic conjugate to $u(r, \phi)$, $v(r, \phi)$ is determined by the formula above. Let $\zeta = re^{i\phi}$. Show that as $r \to 1$ we get

$$v(\phi) = v(r = 0) - \frac{1}{2\pi i} ⨍_0^{2\pi} u(\theta) \frac{e^{i\phi} + e^{i\theta}}{e^{i\phi} - e^{i\theta}} d\theta,$$

this can be rewritten into

$$v(\phi) = v(r = 0) + \frac{1}{2\pi} ⨍_0^{2\pi} u(\theta) \cot\left(\frac{\phi - \theta}{2}\right) d\theta.$$

This formula relates the boundary values, on the circle, between imaginary and real parts of a function $f(z) = u + iv$, which is analytic inside the circle.

Explain what would we get if we tried to establish similar limit ($r \to 1$) for

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \frac{1 - r^2}{[1 - 2r \cos(\phi - \theta) + r^2]} d\theta.$$

4. (AF 7.2.4) Show that the change of variables

$$z = \frac{t - i}{t + i}, \qquad \text{and} \qquad \zeta = \frac{\tau - i}{\tau + i}$$

maps a Cauchy type integral over the real axis τ in the t plane,

$$G(t) = \int_{-\infty}^{\infty} \frac{g(\tau)}{\tau - t} d\tau.$$

to a Cauchy type integral over the unit circle ζ in the z plane

$$F(z) = \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

Note: More precisely, we have $F(z) = \fint_C \frac{f(\zeta)}{\zeta - z} d\zeta$, with some issues at $\zeta = 1$.

5. (AF 7.2.5) Consider the integral

$$U(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta,$$

where u is a real function. This integral is usually referred to as a Schwarz type integral. Establish the following relationship between Schwarz type and Cauchy type integrals

$$U(z) = \frac{1}{2\pi i} \int_C \frac{2u(-i \log \tau)}{\tau - z} d\tau - \int_0^{2\pi} u(\theta) d\theta,$$

where C denotes the unit circle.

(See problem 7.2.6 as well and the connection with the Poisson formula)