Complex analysis II. – Homework 4

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1. (AF 7.4.1) Let C be a finite, closed contour and $f(t)$ a given function satisfying the Hölder condition.

a) Show that the solution of the singular integral equation

$$\frac{1}{\pi i}\fint_C \frac{\phi(\tau)}{\tau - t}d\tau = f(t)$$

is given by

$$\phi(t) = \frac{1}{\pi i}\fint_C \frac{f(\tau)}{\tau - t}d\tau.$$

b) Use (a) to establish the so-called Poincaré–Bertrand formula:

$$f(t) = -\frac{1}{\pi^2}\fint_C \frac{d\tau}{\tau - t}\left(\fint_C \frac{f(\tau')}{\tau' - \tau}d\tau'\right).$$

2. (AF 7.4.2) Define the Hilbert transform of a suitably decaying function $f(x)$ by

$$(Hf)(x) = \frac{1}{\pi}\fint_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x}d\xi.$$

a) Use the results of problem 1 to show that,in the space of suitably decaying functions, $H(Hf(x)) = -f(x)$ (or in shorthand notation we often write $H^2 = -I$).

b) Note that the functions, for real x,

$$\Phi^+(x) = f(x) - i(Hf)(x) \qquad \text{and} \qquad \Phi^-(x) = f(x) + i(Hf)(x)$$

are limits of analytic functions in the upper and lower half complex planes, respectively. Use this fact to establish

$$H[fHg + gHf] = -fg + H(f)H(g).$$

Note. The Hilbert transform can be seen as a convolution of $f(x)$ and $1/x$.

3. (following AF, p. 549) a) Show that if $\Phi^+(z)$ is analytic in the upper half plane, then $\Phi^-(z)$, defined as

$$\Phi^-(z) = \overline{\Phi^+(\bar{z})},$$

is analytic in the lower half plane (this is the Schwarz reflection principle).

b) For the upper half plane we want to find a similar formula as Poisson formula was for the interior of the unit circle, as we have seen in previous homework sets. Hence, we are looking for $\Phi^+(z) = u(x,y) + iv(x,y)$, analytic in the upper half plane, such that on the boundary (the real axis) we have $\operatorname{Re}\Phi^+(x) = u(x,0) = f(x)$ for $x \in \mathbb{R}$. The relation to the imaginary part $\operatorname{Im}\Phi^+(x) = v(x,0) = g(x)$ will be interesting as well.

Using the definition of $\Phi^-(z)$ from (a) show that $\Phi^+(x) + \Phi^-(x) = 2u(x,0) = 2f(x)$ and $\Phi^+(x) - \Phi^-(x) = 2iv(x,0) = 2ig(x)$ for $x \in \mathbb{R}$. Find integral formulae for $\Phi^+(z)$, $u(x,y)$, $v(x,y)$ and $f(x)$ based on $g(x)$ using Plemelj formulae.

c) Chosing $\Psi^+(z) = \Phi^+(z)$ and $\Psi^-(z) = -\Phi^-(z)$, we get $\Psi^+(x) + \Psi^-(x) = 2iv(x,0) = 2ig(x)$ and $\Psi^+(x) - \Psi^-(x) = 2u(x,0) = 2f(x)$. From that, using Plemelj formulae, find $\Psi^+(z)$, $u(x,y)$, $v(x,y)$ and $g(x)$ as integrals based on $f(x)$.

d) Briefly explain connection between $f(x)$, $g(x)$ and Hilbert transform (compare with problem 2b)

4. Evaluate the integral

$$\widehat{\mathrm{rec}}(k) = \fint_{-\infty}^{\infty} \frac{e^{-ikx}}{x}dx, \qquad \text{for } k \in \mathbb{R}.$$

Hint: Use $e^{-ikx} = \cos(kx) - i\sin(kx)$, reducing the problem to evaluating $\int_0^\infty \frac{\sin kx}{x}dx$, p. 240 in AF.

Note: The function $1/x$ is not integrable (it has infinite L_1 norm), its L_2 norm is not finite either. But we can still define its "Fourier transform" using the above principal value integral.