Complex Analysis II. – Homework 5

Due date: November 4, 2014

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1. (AF 3.5.4) Discuss the analytic continuation of the function:

$$\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}, \qquad \text{given by Taylor series for } |z| < 1.$$

2. Show that for the following infinte products we have:

$$\text{a) } \prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \frac{2}{3},$$

$$\text{b) } \prod_{n=2}^{\infty} \cos\frac{\pi}{2^n} = \frac{2}{\pi}.$$

Part b) is in fact the Vieta's product formula mentioned during the lecture:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots$$

3. (AF p. 162) Show that the function given by an infinite product

$$F(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

is analytic for every $z \in \mathbb{C}$ – it is an *entire* function. Where are its zero points?

Show that the infinite product

$$H(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right)$$

diverges for every $z \neq 0$.

Hint: Use the fact $\lim_{w \to 0} \frac{\ln(1+w)}{w} = 1$.

4. (AF 3.6.1) Discuss where the following infinite products converge as a function of $z \in \mathbb{C}$. In cases c) and d) find the values as well:

$$\text{a) } \prod_{n=0}^{\infty} (1 + z^n), \qquad \text{b) } \prod_{n=0}^{\infty} \left(1 + \frac{z^n}{n!}\right),$$

$$\text{c) } \prod_{n=0}^{\infty} \left(1 + z^{2^n}\right), \qquad \text{d) } \prod_{n=1}^{\infty} \left((1 + z^n)(1 - z^{2n-1})\right).$$