Complex analysis II. – Homework 6

Due date: 18. november 2014

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1. Show that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{+\infty} \frac{1}{(z-n)^2}.$$

Hint: a) Show that the series on the right hand side converges uniformly for every compact set not containing $n \in \mathbb{Z}$.

b) Show that the difference between the right and the left side of the equation must be 0; use the fact that both are periodic with period 1 and find their limits for $|y| \to \infty$ (where $z = x + iy$).

2. By integrating the equality from problem 1 show:

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Do not forget to explain why the integration constant must be 0.

3. Show that:

$$\frac{\pi}{\sin \pi z} = \lim_{m \to \infty} \sum_{-m}^{m} (-1)^n \frac{1}{(z-n)}.$$

Hint: Separate the positive and negative parts, use problem 2.

4. Using the expansion of $\cot \pi z$ find the values for the following series (see: Basel Problem, Euler)

$$\sum_{1}^{\infty} \frac{1}{n^2}, \qquad \sum_{1}^{\infty} \frac{1}{n^4}, \qquad \sum_{1}^{\infty} \frac{1}{n^6}.$$

5. Find similar decomposition into fraction series as in problem 3 for the function $1/\cos \pi z$ and show that $\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ (Leibniz formula).