# Complex Analysis II. – Homework 7

Due date: December 2, 2014

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**1.** By a direct computation show that
$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}.$$
Explain the connection with the result of Problem 3.

**2.** (AF pp. 226 – 230) Find the contour integral
$$I=\frac{1}{2\pi i}\oint_C \pi\cot\pi\zeta\left(\frac{1}{\zeta}-\frac{1}{\zeta-z}\right)d\zeta,\qquad (z\neq 0,\pm1,\pm2,\pm3,\dots),$$
where $C$ is a boundary of a rectangular domain $x\in[-N-\frac{1}{2},N+\frac{1}{2}]$, $y\in[-N,N]$ and $z$ lies inside this domain.

Deduce as a consequence
$$\pi\cot\pi z=\frac{1}{z}+\sum_{n\neq 0}\left(\frac{1}{z-n}+\frac{1}{n}\right).$$

**3.** Show that
$$\sin\pi z=\pi z\prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{z/n}$$
by logarithmic differentiation of both sides and using the result of Problem 2 or HW 6.

**4.** (AF 3.6.9) Let $f(z)$ have simple poles at $z=z_n\in D_N$, $n=1,2,3,\dots,N$ with strengths $a_n$ and is analytic everywhere else. Show by contour integration that
$$\frac{1}{2\pi i}\oint_{C_N}\frac{f(\zeta)}{\zeta-z}d\zeta=f(z)+\sum_{n=1}^{N}\frac{a_n}{z_n-z},$$
where $C_N$ is the boundary of $D_N$.

Evaluate at $z=0$ to obtain
$$\frac{1}{2\pi i}\oint_{C_N}\frac{zf(\zeta)}{\zeta(\zeta-z)}d\zeta=f(z)-f(0)+\sum_{n=1}^{N}a_n\left(\frac{1}{z_n-z}-\frac{1}{z_n}\right).$$

Assuming that $f(z)$ is bounded for large $z$ show that the left-hand side vanishes for the circular domain $D_N$ with radius $R_N$ as $R_N\to\infty$.

Similarly, find corresponding contour integral which leads to
$$f(z)-f(0)-zf'(0)-\dots-\frac{z^k f^{(k)}(0)}{k!}+\sum_{n=1}^{N}a_n\left(\frac{1}{z_n-z}-\frac{1}{z_n}-\frac{z}{z_n^2}-\dots-\frac{z^k}{z_n^{k+1}}\right).$$
What is the necessary condition on $f(z)$ for large $z$ for contour integral on the left-hand side to vanish as $R_N\to\infty$? I.e. what should be the behavior of $f(z)$ of genus $k$ for large $z$?

*Note:* Terms which we included into the sum of partial fractions are precisely the correction terms from Mittag–Leffler expansions.