Complex analysis II. – Homework 8

Due date: December 9, 2014

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1. (Stein, Shakarchi, pp. 118–119) Prove the Poisson formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Hint: Using residues show

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} \frac{f(z)}{e^{2\pi iz}-1}dz - \int_{L_2} \frac{f(z)}{e^{2\pi iz}-1}dz,$$

where L_1, L_2 are lines obtained by shifting the real line up and down by b. Use $\frac{1}{w-1} = w^{-1}\sum_{n=0}^{\infty} w^{-n}$ for $|w|>1$, and $\frac{1}{w-1} = -\sum_{n=0}^{\infty} w^n$ for $|w|<1$ to express integrals along L_1 and L_2 as sums of values of $\hat{f}(n)$.

2. The theta function is defined for $t>0$ as $\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$. Using Poisson formula for $e^{-\pi t x^2}$ deduce the functional equation

$$\vartheta(t) = t^{-1/2}\vartheta(1/t), \qquad \text{for } t>0.$$

Hint: Start with the Fourier transform of $e^{-\pi x^2}$ and make a change of variables $x \mapsto t^{1/2}(x+a)$.

3. Prove that for $\mathrm{Re}(s)>1$,

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{x^{s-1}}{e^x-1}dx.$$

Hint: Use $1/(e^x-1) = \sum_{n=1}^{\infty} e^{-nx}$.

4. (Stein, Shakarchi, 6.16, pp. 178–179) Writing

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^1 \frac{x^{s-1}}{e^x-1}dx + \frac{1}{\Gamma(s)}\int_1^{\infty} \frac{x^{s-1}}{e^x-1}dx$$

for $\mathrm{Re}(s)>1$, show that the second integral defines an entire function, while

$$\int_0^1 \frac{x^{s-1}}{e^x-1}dx = \sum_{m=0}^{\infty} \frac{B_m}{m!(s+m-1)},$$

where B_m denotes m^{th} Bernoulli number defined by

$$\frac{x}{e^x-1} = \sum_{m=0}^{\infty} \frac{B_m}{m!}x^m.$$

Since $z/(e^z-1)$ is holomorphic for $|z|<2\pi$, deduce $\limsup_{m\to\infty} |B_m/m!|^{1/m} = 1/2\pi$, and investigate the continuation of the first integral divided by $\Gamma(s)$ for $\mathrm{Re}(s)\le 1$.

5. Similarly as in Problem 4, HW 6 show

$$\sum_{1}^{\infty} \frac{1}{n^{2k}} = 2^{2k-1}\frac{B_{2k}}{(2k)!}\pi^{2k}.$$

Hint: Find the Taylor series of $\tan(z)$ for $z=0$ using Bernoulli numbers.