Complex analysis II. – Homework 9

Due date: December 16, 2014

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1. (Stein, Shakarchi, 7.1, pp. 199–200) Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that the partial sums

$$A_n = a_1 + \cdots + a_n$$

are bounded. Prove that the *Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for $\mathrm{Re}(s) > 0$ and defines a holomorphic function in this half-plane.

Hint: Use summation by parts to compare the original (non-absolutely convergent) series to the (absolutely convergent) series $\sum A_n(n^{-s} - (n+1)^{-s})$. An estimate for the term in parentheses is provided by the mean value theorem.

2. (Stein, Shakarchi, 7.2) Link the multiplication of Dirichlet series with the divisibility properties of their coefficients.

(a) Show that if $\{a_m\}$ and $\{b_k\}$ are two bounded sequences of complex numbers then

$$\left(\sum_{m=1}^{\infty} \frac{a_m}{m^s}\right)\left(\sum_{k=1}^{\infty} \frac{b_k}{k^s}\right) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} \qquad \text{where } c_n = \sum_{mk=n} a_m b_k.$$

The above series converge absolutely when $\mathrm{Re}(s) > 1$.

(b) Prove as a consequence that one has

$$(\zeta(s))^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \qquad \text{and} \qquad \zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s}$$

for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s-a) > 1$, respectively. Here $d(n)$ equals the number of divisors of n, and $\sigma_a(n)$ is the sum of the a^{th} powers of divisors of n. In particular, one has $\sigma_0(n) = d(n)$.

3. (Stein, Shakarchi, 7.3) Consider the Dirichlet series for $1/\zeta$.

(a) Prove that for $\mathrm{Re}(s) > 1$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where $\mu(n)$ is the *Möbius function* defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 p_2 \dots p_k, \text{ and the } p_j \text{ are distinct primes}, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $\mu(nm) = \mu(n)\mu(m)$ whenever n and m are relatively prime.

Hint: Use Euler product formula for $\zeta(s)$.

(b) Show that

$$\sum_{k|n} \mu(k) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise}. \end{cases}$$

4. (Stein, Shakarchi, 7.4) Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $a_n = a_m$ if $n \equiv m \mod q$ for some positive integer q. Define the *Dirichlet L-series* associated to $\{a_n\}$ by

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad \text{for } \mathrm{Re}(s) > 1.$$

Also, with $a_0 = a_q$, let

$$Q(x) = \sum_{m=0}^{q-1} a_{q-m} e^{mx}.$$

Show, as in problems 3 and 4 in HW 8, that

$$L(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{Q(x)x^{s-1}}{e^{qx} - 1} dx, \qquad \text{for } \mathrm{Re}(s) > 1.$$

Prove as a result that $L(s)$ is continuable into the complex plane, with the only possible singularity a pole at $s = 1$. In fact, $L(s)$ is regular at $s = 1$ if and only if $\sum_{m=0}^{q-1} a_m = 0$.

5. (Stein, Shakarchi, 7.7) Show that the function

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is real when s is real, or when $\mathrm{Re}(s) = 1/2$.